Dynamic monopoly pricing when competing with new experience substitutes*

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ABSTRACT

In this paper, I study the dynamic problem of a monopolistic seller who suddenly finds the dominant market position of her product challenged by the appearance of a competitively supplied substitute of uncertain value for her customers. I construct Markov perfect equilibria with and without price discrimination for the case of two types of consumers who may learn their idiosyncratic valuation stochastically as they try out the new product. If the seller can tailor her pricing policy to individual consumer experience, the equilibrium is efficient. Without the ability to charge different prices, dynamic inefficiencies arise, since consumers experiment too much. However, the asymptotic outcomes are almost surely efficient.

KEYWORDS: Bayesian learning; Dynamic pricing; Experience goods; Price discrimination;

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1. Introduction

In this paper, I study the market dynamics triggered by the introduction of a new product. For example, consider the market for operating systems in the late 1990’s or the market for mobile devices in the late 2000’s. In each case a monopoly supplier of an established product – Microsoft Windows in one case and Apple iPhone in the other – confronts the introduction of a new product (Linux distributions or Android phones). The new products are based on open source software and, as a result, are supplied by a large number of competing firms. Thus, to a first approximation, the new product is supplied by a competitive industry.

Similar scenarios take place with the development of other technologies. For example, the market power of traditional phone companies has been weakened by the advent of Skype and other “Voice over IP” (VoIP) protocols. Any computer with internet access can connect to standard phones through one of the many VoIP services available, providing users with a cheaper alternative for making calls.

In these examples the level of penetration of the new product varies across different market segments. For instance, Linux was highly successful in the server and technical computing segments, where it became the leading system, but did not fare as well with home users. In the VoIP example, computer savvy and frequent long-distance callers are more likely to adopt the new technology than the average client.

In general, we can expect heterogeneous diffusion patterns when the new product is relatively better suited to fulfill the needs of particular classes of consumers. However, the new product’s ability to satisfy different consumers is typically not clear at the outset. Expectations may change over time as technology evolves, new possibilities become available and consumers learn more about the strengths and weaknesses of the new product for their purported uses. As a result, consumers’ experiences with the new product play an important role in shaping their preferences and product choices.
However, the incentives to experiment also depend critically on the price path that the established product is expected to follow. Since this price is strategically controlled by the seller, she could deter consumer experimentation by charging lower prices, if she so wished.

Comparable market interactions may take place after the expiration of a patent or the passage of legislation softening barriers to entry in an industry. For example, in 2002 the Congress of Argentina passed a law promoting generic drugs which forbids doctors to prescribe pharmaceuticals by its brand name. This abruptly increased the competition faced by traditional laboratories at multiple points in their product lines, as patients massively became aware of a choice between original brand name drugs and cheaper but less familiar generic alternatives.¹

All these situations share a common structure. A monopolistic seller is suddenly exposed to increased competition and might react strategically by changing prices. On the demand side, consumer choices are affected by the seller’s actions and this determines the amount of experimentation. To study this class of interactions, I consider a simple dynamic stochastic game in continuous time. At every date, the seller sets a price for her product. After seeing this price, consumers can freely choose between two alternatives. They can either buy the established product from the seller or try the new product, which is supplied competitively. When consumers choose the new product, their experiences provide a continuous flow of information, which affects their willingness to pay for the new product.

Consumers are grouped in different market segment. I assume that the average experience of each segment is public information and is independent of the experience of other segments. Although these are admittedly bold assumptions, they seem a reasonable approximation to some real-world situations. For instance,

¹ Generics are usually required to be statistically as safe and effective as the originals, but they do not need to be identical. This can raise efficacy concerns among patients. Moreover, the use of generics might entail different legal rights. For instance, the Supreme Court of the United States ruled in 2011 that generic manufacturers cannot be held responsible for failing to alert patients to problems with their drug.
segment-specific product reviews, ratings and market share figures are easily available and frequently updated in many cases. In those cases, they are likely to influence average individual decisions within the corresponding segment. Moreover, when segments are very different in nature, information about how the substitute performed for other segments should have little relevance beyond forecasting future market conditions.

For simplicity, I limit the analysis to two market segments populated by a continuum of identical consumers. As a benchmark, I solve the problem of a benevolent planner. The efficient product choice strategies depend only on individual consumer beliefs and have a cutoff form. That is, it is efficient for consumers to adopt the new product if and only if they believe is superior to the established product with sufficiently high probability.

Then, I turn to equilibrium analysis. In general, when consumers experiment with the new product, they obtain an option value because they might choose to switch back to the established product in the future, if their interim experience is unsatisfactory. Not surprisingly, when the seller can charge different prices to different consumers, she is able to extract all this option value by adjusting prices appropriately. As a consequence, the incentives of the seller mimic those of the planner and the resulting equilibrium is efficient.

In contrast, dynamic inefficiencies arise when price discrimination is not allowed and expected valuations across market segments differ appropriately. In equilibrium, the seller is reluctant to reduce prices, distorting the incentives of those consumers who are more optimistic about the new product. As a consequence, these relatively optimistic consumers may refuse to buy from the seller, even when the planner would prescribe it. This dynamic inefficiency might be very significant and persistent, depending on parameter values. Nevertheless, I show that, if ex-ante the new product can be better than the established one, consumers end up choosing products efficiently after sufficiently long time with probability 1.
The analysis in this paper is performed under the assumption that the seller cannot commit to a pricing strategy. Because consumers are small and lack the ability to individually affect the aggregate outcome, allowing commitment does not make any difference. Moreover, even if we allowed non-negligible consumers, commitment would not affect the equilibrium when price discrimination is feasible because the non-commitment equilibrium would still reward the seller with the maximum possible payoff in any mechanism in which consumers participate voluntary. Without price discrimination, the seller might benefit from commitment. However, she would still be forced to leave rents to inframarginal consumers. It follows that even the best equilibrium payoff with commitment but without price discrimination must be inferior to the equilibrium payoff obtained when the seller has the ability to charge different prices.

1.1 Related literature

This paper is related to Bergemann and Välimäki (1997), who study product diffusion in a duopoly with differentiated products. In their model, the value of the new product is the same for all consumers, but there is horizontal differentiation to accommodate duopolistic competition. Because my focus is on defensive pricing rather than product diffusion, I extend their model by allowing idiosyncratic valuations and simplify it by assuming a competitive supply of the new product. On one hand, allowing heterogeneity in the object of uncertainty is important in markets where multiple segments can be distinguished or where personal experiences vary greatly. Moreover, it is a crucial dimension to effectively discuss dynamic price discrimination in experimentation contexts. On the other hand, simplifying the market structure allows me to solve the equilibrium in closed form with a positive discount factor, while Bergemann and Välimäki (1997) limit their analysis to the case of zero time-discounting for tractability reasons.
The analytics of the present paper traces back to Bolton and Harris (1999), who were the first to study strategic experimentation in a continuous time model. However, their focus is quite different, as their model abstracts from strategic pricing and features symmetric players individually choosing whether or not to experiment in order to collectively learn about a common uncertain valuation.

Finally, the paper is also connected with the traditional statistical literature on bandit problems, which was introduced to economic analysis by Rothschild (1974) who studied the problem of choosing prices when the demand curve is unknown and can only be learned through experience. However, the bandits faced by consumers in my model have endogenous payoffs which are strategically controlled by the seller through prices.

1.2 Outline

The rest of the paper is organized as follows. Section 2 setups the model. Section 3 studies efficient product choices. Section 4 constructs an efficient Markov perfect equilibrium when the seller can charge different prices to different consumers. Section 5 constructs a Markov perfect equilibrium for the more interesting case in which price discrimination is not feasible. The equilibrium exhibits dynamic inefficiencies in the form of over-experimentation with the new product. Section 6 briefly discusses some natural extensions and modifications of the basic model. I offer some final remarks in Section 7. All the proofs are contained in the Appendix.
2. Model

There are three kinds of players: two groups of consumers, represented by Ann and Bob, and a seller offering a non-storable good. The players interact in continuous time with an infinite horizon and are risk neutral expected discounted utility maximizers with common discount rate \( r > 0 \). At each date \( t \geq 0 \), the seller offers a unit of her product to each market segment \( i \in \{A, B\} \) at a price \( p_t^i \geq 0 \). After seeing the corresponding price, each consumer decides whether to buy from the seller or try a new substitute which is competitively supplied at a zero price. Ann and Bob know the seller’s product, but are uncertain about their valuation of the new product. The seller bears no costs.

Let \( Z^A = \{Z_t^A\} \), \( Z^B = \{Z_t^B\} \) be two independent standard Brownian motions. Let \( q_t^A \in [0,1] \) indicate the fraction of consumers in Ann’s segment buying the seller’s product. Her utility is a stochastic process \( X^A = \{X_t^A\} \) which evolves according to the following stochastic differential equation (SDE):

\[
dX_t^A = [q_t^A(\mu^* - p_t^A) + (1 - q_t^A)\mu^A]dt + \sqrt{1 - q_t^A\sigma}dZ_t^A, \tag{1}
\]

where \( \mu^* \) represents the known flow value of the seller’s product and \( \mu^A \) is an unknown parameter representing Ann’s idiosyncratic flow value of the new product. The parameter \( \sigma > 0 \) is known and measures the level of noise associated with the new product. Without loss of generality, assume that \( \mu^A \in \{0,1\} \). Moreover, suppose that \( \mu^* \in (0,1) \), so there is an actual efficiency trade-off between the two products. The case \( \mu^* \geq 1 \) is simpler, as the seller’s product dominates the new product irrespective of beliefs (for a brief analysis, see Section 6.4).

I assume that the processes \( X^A \) and \( X^B \) are publicly observable. Let \( \mathcal{F}_t^A \) be the filtration generated by \( X^A \) and \( \theta_t^A \) the probability Ann assigns to \( \mu^A = 1 \) given the information she has up to time \( t \).
From standard filtering theory we know that

$$d\theta_t^A = (1 - q_t^A) \sqrt{\nu(\theta_t^A)} \left( \frac{\mu^A - \theta_t^A}{\sigma} \right) dt + \sqrt{1 - q_t^A} \nu(\theta_t^A) dZ_t^A,$$

(2)

where $\nu(\theta) := \theta^2(1 - \theta)^2 / \sigma^2$. Hence, if $\mu^A = 1$ and Ann tries the new product, the process $\theta^A$ will have positive drift and she will tend to become more optimistic about $\mu^A$ over time. The magnitude of this drift is decreasing in $\sigma$.

Note that $Z^A$ cannot be adapted to $\mathcal{F}_t^A$. If it was, equation (2) would allow Ann to infer $\mu^A$. Instead, equation (2) represents the evolution of $\theta^A$ from the perspective of an outsider who knew $\mu^A$. From Ann’s perspective, $\theta^A$ evolves according to

$$d\tilde{\theta}_t^A = \sqrt{(1 - q_t^A)\nu(\theta_t^A)} d\tilde{Z}_t^A.$$

(3)

where $\tilde{Z}^A$ is Ann’s “innovation process”. This is the evolution of beliefs that players take into account to compute the value of different strategies. Note that $\theta^A$ is a $\{\mathcal{F}_t^A\}$-martingale. As for Bob, we define $q^B, X^B, \{\mathcal{F}_t^B\}, \{\tilde{\theta}_t^B\}$ and $\mu^B$ similarly. In what follows, whenever I define a quantity for Ann, consider an analogous quantity automatically defined for Bob. For simplicity, I will focus on the symmetric case and assume that $\mu^*$ and $\sigma$ are the same for both market segments.

Since I am mostly interested in the case in which price discrimination is not feasible, all the following definitions correspond to that case. Adapting the definitions to the case with price discriminations should be obvious. At each point in time $t \geq 0$, the seller sets a price for her product. She does so knowing exactly the beliefs held by Ann and Bob. Formally, let $\{\mathcal{F}_t\}$ denote the filtration generated by $X \equiv (X^A, X^B)$ and let $\tilde{\mathcal{P}}$ be the set of stochastic processes taking values in $\mathbb{R}_+$ which are progressively measurable w.r.t. $\{\mathcal{F}_t\}$. A pricing strategy for the seller is an element $p \in \tilde{\mathcal{P}}$. Simultaneously, consumers choose which product to buy considering both their experience and the current price offered to them. Since the decision is binary and Ann’s experienced utility decreases with the price paid, we can represent
her strategies with the maximum price she is willing to pay for the established good. Formally, a purchasing strategy for Ann is a stochastic process \( \bar{p}_t^A \in \bar{P} \), just as in the case of the seller. The interpretation is that Ann buys from the seller if and only if \( p_t^A \leq \bar{p}_t^A \) and experiments otherwise.

We are now in position to define payoffs. Given a strategy profile \((\bar{p}, \bar{p}^A, \bar{p}^B) \in \bar{P} \times \bar{P} \times \bar{P}\), the expected discounted revenue of the seller at time \( t \) is given by

\[
R_t(\bar{p}, \bar{p}^A, \bar{p}^B) := \mathbb{E}\left\{ \int_t^\infty e^{-r(t-\tau)} (1\{\bar{p}_\tau \leq \bar{p}_t^A\} \bar{p}_t^A + 1\{\bar{p}_\tau \leq \bar{p}_t^B\} \bar{p}_t^B) d\tau \left| F_t \right. \right\}.
\]  

The expected discounted utility of Ann is

\[
\bar{U}_t^A(\bar{p}, \bar{p}^A, \bar{p}^B) := \mathbb{E}\left\{ \int_t^\infty e^{-r(t-\tau)} (1\{\bar{p}_\tau \leq \bar{p}_t^A\} (\mu^* - \bar{p}_t^A) + 1\{\bar{p}_\tau > \bar{p}_t^A\} \mu^A) d\tau \left| F_t \right. \right\}.
\]  

Note that, by definition, we have \( \mathbb{E}[\mu^A|F_\tau, \bar{p}_\tau] = \mathbb{E}[\mu^A|F_\tau] = \theta_t^A \). Hence, by the law of iterated expectations, we can write

\[
\bar{U}_t^A(\bar{p}, \bar{p}^A, \bar{p}^B) = \mathbb{E}\left\{ \int_t^\infty e^{-r(t-\tau)} (1\{\bar{p}_\tau \leq \bar{p}_t^A\} (\mu^* - \bar{p}_t^A) + 1\{\bar{p}_\tau > \bar{p}_t^A\} \theta_t^A) d\tau \left| F_t \right. \right\}.
\]

It follows that all the payoff-relevant non-strategic elements are encoded in the beliefs described by the Markov process \((\theta^A, \theta^B) = ((\theta_t^A, \theta_t^B))\). This suggests defining Markov strategies taking \((\theta_t^A, \theta_t^B)\) as the state. In this way, a Markov pricing strategy is a measurable function \( p: [0,1]^2 \to \mathbb{R}_+ \) such that \( \bar{p}_t := p(\theta_t) \) defines a pricing strategy. A Markov purchasing strategy for Ann is a measurable function \( \bar{p}^A: [0,1]^2 \to \mathbb{R}_+ \) such that \( \bar{p}_t^A := \bar{p}^A(\theta_t) \) defines a purchasing strategy. Let \( \mathcal{P} \) denote the set of all Markov strategies for the seller, Ann and/or Bob. We can now write payoffs as time-invariant functions of Markov strategies and the state. Thus, the revenue of the seller in state \( \theta \) is

\[
R(\theta, p, \bar{p}^A, \bar{p}^B) := \mathbb{E}\left\{ \int_0^\infty e^{-rt} (1\{p(\theta_t) \leq \bar{p}^A(\theta_t)\} + 1\{p(\theta_t) \leq \bar{p}^B(\theta_t)\}) p(\theta_t) dt \left| \theta_0 = \theta \right. \right\}.
\]
The expected utility of Ann in state $\theta$ can be written

$$U^A(\theta, p, \tilde{p}^A, \tilde{p}^B) = \mathbb{E} \left\{ \int_0^\infty e^{-rt} (1\{p(\theta) \leq \tilde{p}^A(\theta)\})(\mu^* - p(\theta) - \theta^A) + \theta^A) \, dt \middle| \theta_0 = \theta \right\}. \quad (8)$$

Note that $U^A(\theta, p, \tilde{p}^A, \tilde{p}^B)$ in principle depends on $\theta^B$, since Bob’s beliefs influence present and future prices, therefore affecting Ann’s current payoff.

An equilibrium is a strategy profile $(\bar{p}, \bar{p}^A, \bar{p}^B) \in \bar{P} \times \bar{P} \times \bar{P}$ such that, for every $t$, the following equalities hold almost surely:

$$\bar{r}_t(\bar{p}, \bar{p}^A, \bar{p}^B) = \sup \{ \bar{r}_t(\bar{p}, \bar{p}^A, \bar{p}^B) \middle| \bar{p} \in \bar{P} \} \quad (9)$$

$$\bar{u}_t^A(\bar{p}, \bar{p}^A, \bar{p}^B) = \sup \{ \bar{u}_t^A(\bar{p}, \bar{p}^A, \bar{p}^B) \middle| \bar{p} \in \bar{P} \} \quad (10)$$

$$\bar{u}_t^B(\bar{p}, \bar{p}^A, \bar{p}^B) = \sup \{ \bar{u}_t^B(\bar{p}, \bar{p}^A, \bar{p}^B) \middle| \bar{p} \in \bar{P} \}. \quad (11)$$

A Markov perfect equilibrium (MPE) is a Markov strategy profile $(p, \bar{p}^A, \bar{p}^B) \in P \times \bar{P} \times \bar{P}$ such that the induced strategies on $\bar{P} \times \bar{P} \times \bar{P}$ form an equilibrium.
3. Efficiency

In this section, I analyze the problem of a benevolent planner who possesses the same information as the players and tries to maximize total social surplus. Since there is no loss of generality in restricting attention to Markov strategies, we can define maximal total surplus at state $\theta$ by

$$
S(\theta) := \sup \{R(\theta,p,\bar{p}^A,\bar{p}^B) + U^A(\theta,p,\bar{p}^A,\bar{p}^B) + U^B(\theta,p,\bar{p}^A,\bar{p}^B) | p, \bar{p}^A, \bar{p}^B \in \mathcal{P} \}
$$

$$
= \sup \{U^A(\theta,0,\bar{p}^A,\bar{p}^B) + U^B(\theta,0,\bar{p}^A,\bar{p}^B) | \bar{p}^A, \bar{p}^B \in \mathcal{P} \},
$$

(12)

where we used the fact that production has zero cost (so it involves no efficiency issue) and prices paid are only transfers from consumers to the firm. Moreover, since the planner can tailor the purchasing strategy of each consumer to its individual belief state, we can write:

$$
S(\theta) = S^A(\theta^A) + S^B(\theta^B),
$$

(13)

where, for each $i \in \{A, B\}$, we define the individual surplus as:

$$
S^i(\theta^i) := \sup_{\bar{p}^i \in \mathcal{P}} \left\{ \mathbb{E} \left[ \int_0^\infty e^{-\tau t} \left( 1\{\bar{p}^i(\theta^i) = 0\} \mu^* + 1\{\bar{p}^i(\theta^i) > 0\} \theta^i_t \right) dt \Bigg| \theta^i_0 = \theta^i \right] \right\}.
$$

(14)

The fact that total surplus is separable allows us to study each allocation problem individually. For the rest of this section, I will focus on Ann’s problem. The RHS of (14) defines a stochastic control problem. It is natural to resort to the dynamic programming paradigm and seek a recursive expression for the value function $S^i$ by writing down the associated Hamilton-Jacobi-Bellman (HJB) equation:

$$
rS^A = \max \left\{ \mu^*, \theta^A + \frac{1}{2} v(\theta^A) \left( \frac{d^2 S^A}{d\theta^A} \right) \right\},
$$

(15)

where we accept to substitute $d^2 S^A / d\theta^A$ for the side-derivative at switching points. Note that, even with this proviso, equation (15) only makes sense if the value function $S$ is smooth enough and I haven’t shown that so far. However, it is possible
to use the HJB equation to find a sufficiently smooth candidate and later verify that the candidate actually solves the planner’s problem. Under the assumption that $S^A$ satisfies equation (15), Ann should experiment the new product if

$$\theta^A + \frac{1}{2} \nu(\theta^A) \left( \frac{d^2 S^A}{d \theta^A \theta^2} \right) > \mu^*.$$  

(16)

If this condition does not hold, Ann should keep consuming the seller’s product. That is, we expect the experimentation region to be an interval of the form $(\theta^*, 1]$ for some belief cutoff $\theta^* \in (0,1)$. In that region, the HJB equation reads

$$r S^A = \theta^A + \frac{1}{2} \nu(\theta^A) \left( \frac{d^2 S^A}{d \theta^A \theta^2} \right).$$  

(17)

This is a second-order linear ordinary differential equation (ODE) with variable coefficients. We solve it subject to the following boundary conditions:

1) (Absorption at the top) \hspace{1cm} S^A(1) = 1/r 
2) (Value matching) \hspace{1cm} S^A(\theta^*) = \mu^*/r 
3) (Smooth pasting) \hspace{1cm} (S^A)'(\theta^*) = 0. 

(18)

Note that, since the cutoff $\theta^*$ is not known, the equations in (17) and (18) define a free-boundary problem. Condition 1 states the intuitive fact that, when Ann is (almost) sure that $\mu^A = 1 > \mu^*$ (i.e. if $\theta^A = 1$), then she should consume the new product to maximize social surplus. Condition 2 says that, at $\theta^*$, Ann must be indifferent between the two products, taking into account the total expected value of experimentation (i.e. including the option of switching back to the seller’s product in the future). Finally, condition 3 requires $S^A$ to be continuously differentiable at the cutoff and is a standard condition in optimal stopping problems of economic interest.
To state the solution to this problem, define $H: [0, 1] \to \mathbb{R}$, $\alpha$ and $\beta$ by

$$H(z) := z^\alpha (1 - z)^\beta \quad \alpha := \frac{1 - \sqrt{1 + 8r\sigma^2}}{2} < 0 \quad \beta := \frac{1 + \sqrt{1 + 8r\sigma^2}}{2} > 1. \quad (19)$$

Using these definitions, the next result solves the free-boundary problem above and establishes that the solution provided characterizes the efficient allocation:

**Proposition 1.** The efficient allocation is to have consumer $i \in \{A, B\}$ experimenting if and only if $\theta^i > \theta^*$, where the cutoff is given by:

$$\theta^* := \frac{\alpha \mu^*}{\mu^* - \beta} \in (0, \mu^*). \quad (20)$$

The maximal social surplus on consumer $i$ satisfies:

$$rS_i(\theta^i) = \begin{cases} 
\mu^* & \theta \leq \theta^* \\
\theta^i - (\theta^* - \mu^*) \frac{H(\theta^i)}{H(\theta^*)} & \theta > \theta^*.
\end{cases} \quad (21)$$

Equations (20) and (21) give an explicit formula for the efficient cutoff and for maximal total social surplus, respectively. Differentiating, we can obtain definite comparative statics on $\theta^*$ with respect to $\mu^*$, $r$ and $\sigma$:

$$\frac{\partial \theta^*}{\partial \mu^*} > 0 \quad \frac{\partial \theta^*}{\partial r} > 0 \quad \frac{\partial \theta^*}{\partial \sigma} > 0. \quad (22)$$

Hence, we get the intuitive result that a higher valuation for the seller’s product implies a higher experimentation cutoff. Moreover, the cutoff is also increasing in $r$ and $\sigma$. This is also intuitive since, as the discount rate or the level of noise increases, efficiency requires better expectations about the new product in order to experiment (in the first case because time is more valuable, in the second because experience is less informative).
We can also show that $S^i(\theta) > 0$ for all $\theta > \theta^*$, so $S^i$ is non-decreasing in $\theta$. The following figures plot the maximal social surplus as a function of beliefs:

**Figure 1.** Maximal total surplus for Ann.

**Figure 2.** Maximal total surplus for Ann and Bob.
The following figure illustrates the separability of the planner’s problem and the stochastic dynamics implied by efficiency.

![Figure 3](image)

**Figure 3.** Efficient experimentation regions
(thick black lines and dots indicate rest points).

When both Ann and Bob are pessimistic about the new product, experimentation is inefficient and beliefs remain at rest. If Ann’s belief exceeds $\theta^*$, then the planner will have her consuming the new product and the dynamic system can move in the horizontal direction. If $\mu^A = 1$, as Ann experiments, she might asymptotically learn her valuation or she might end up switching back to the seller when $\theta^t_A = \theta^*$. The latter will happen almost surely if $\mu^A = 0$. Symmetric considerations apply to Bob.
4. Equilibrium with price discrimination

In this section, I construct a MPE for the case in which the seller can offer Ann and Bob different prices. Suppose the strategy profile \((\hat{\phi}, \bar{\phi}) = (\hat{\phi}^A, \hat{\phi}^B, \bar{\phi}^A, \bar{\phi}^B)\) is an MPE and define equilibrium value functions \(\Pi, V^A\) and \(V^B\) as follows:

\[
\Pi(\theta) := R(\theta, \hat{\phi}, \bar{\phi}) \quad V^A(\theta) := U^A(\theta, \hat{\phi}, \bar{\phi}) \quad V^B(\theta) := U^B(\theta, \hat{\phi}, \bar{\phi}).
\] (23)

If we allow for price discrimination, the problem of the seller becomes separable because both Ann and Bob are marginal buyers. In order to maximize profits, the seller will have to set prices which make each consumer indifferent whenever they choose to buy from her. Because no individual consumer can affect the aggregate level of experimentation in its segment, \(\hat{\phi}^A(\theta) \leq \bar{\phi}^A(\theta)\) implies

\[
\hat{\phi}^A(\theta) = \bar{\phi}^A(\theta) = \mu^* - \theta^A.
\] (24)

As a result, Ann’s equilibrium value satisfies the following HJB equation:

\[
rv^A(\theta) = \theta^A + \frac{1}{2} v(\theta^A) \frac{\partial^2 v^A(\theta)}{\partial \theta^A^2} 1\{\hat{\phi}^A(\theta) > \bar{\phi}^A(\theta)\}
\]

\[
+ \frac{1}{2} v(\theta^B) \frac{\partial^2 v^A(\theta)}{\partial \theta^B^2} 1\{\hat{\phi}^B(\theta) > \bar{\phi}^B(\theta)\}.
\] (25)

I am seeking a MPE in which Ann’s value depends upon Bob’s belief only through its effect on prices. Since we are allowing price discrimination, we can expect the price offered to Ann to be independent of Bob’s belief. Thus, I will assume that \(V^A(\theta)\) does not depend on \(\theta^B\) and construct a MPE satisfying this assumption. Then, the HJB equation for Ann simplifies to

\[
r v^A(\theta) = \theta^A + \frac{1}{2} v(\theta^A) \frac{\partial^2 v^A(\theta)}{\partial \theta^A^2} 1\{\hat{\phi}^A(\theta) > \bar{\phi}^A(\theta)\}.
\] (26)
Equation (26) holds for all $\theta^A \in [0,1]$ since, when Ann purchases the seller’s product, she is a marginal buyer and therefore gets exactly what she would get had she been experimenting. In particular, when $\theta^A = 0$, Ann must get zero in any equilibrium. The unique solution to equation (26) which satisfies $V^A(0, \theta^B) = 0$ is

$$V^A(\theta) = \frac{\theta^A}{r}. \quad (27)$$

Note that this solution is independent of the value of $\theta^B$. Moreover, the seller appropriates all of Ann’s option value by tailoring prices to Ann’s experience with the new product.

Let $\tilde{\theta}^A$ be the maximal value of $\theta^A$ at which Ann buys from the seller in equilibrium. We expect Ann to buy from the seller for every $\theta^A \in [0, \tilde{\theta}^A]$ at price $\hat{p}^A(\theta) = \mu^* - \theta^A$. For $\theta^A > \tilde{\theta}^A$, we can have the seller offering any price $\hat{p}^A(\theta) \geq \hat{p}^A(\tilde{\theta}^A) = \mu^* - \tilde{\theta}^A$ since the seller is not interested in having her price accepted.

It remains to determine the cutoff $\tilde{\theta}^A$ and the seller’s equilibrium profits. These are determined by the seller, who chooses when to stop offering Ann a deal which renders her indifferent between products. Formally, the seller’s profit from doing business with Ann satisfies

$$r \Pi^A(\theta^A) = \max \left\{ \mu^* - \theta^A, \frac{1}{2} v(\theta^A) \frac{d^2 \Pi^A(\theta^A)}{d \theta^A^2} \right\}. \quad (28)$$

Note that, defining $Y^A(\theta^A) := \Pi^A(\theta^A) + \theta^A/r$, we can write

$$rY^A = \max \left\{ \mu^*, \theta^A + \frac{1}{2} v(\theta^A) \frac{d^2 Y^A}{d \theta^A^2} \right\}. \quad (29)$$

By inspection, it is clear that this problem is exactly the problem of the planner represented in equation (15). It follows from the analysis in the previous section that $\tilde{\theta}^A = \theta^*$. Hence, the Markov strategy profile we are constructing is efficient.
Solving equation (28) on the region $\theta^A > \theta^*$ and using the boundary condition $\Pi^A(\theta^*) = \mu^* - \theta^*$, we can see that the profit the seller extracts from Ann is:

$$r\Pi^A(\theta^A) = \begin{cases} 
(\mu^* - \theta^*) \left( \frac{H(\theta^A)}{H(\theta^*)} \right) & \theta^A > \theta^* \\
\mu^* - \theta^A & \theta^A \leq \theta^*. 
\end{cases} \quad (30)$$

The following result summarizes the previous analysis:

**Proposition 2.** The Markov strategy profile $(\hat{p}^A, \hat{p}^B, \bar{p}^A, \bar{p}^B)$ defined by

$$\hat{p}^i(\theta) := \mu^* - \min\{\theta^i, \theta^*\} \quad \bar{p}^i(\theta) := \mu^* - \theta^i \quad (31)$$

is a MPE and prescribes an efficient product choice in every state.

The choices induced by $(\hat{p}^A, \hat{p}^B, \bar{p}^A, \bar{p}^B)$ are easily seen to be efficient since

$$\mathbb{1}\{\hat{p}^i(\theta) \leq \bar{p}^i(\theta)\} = \mathbb{1}\{\theta^i \leq \min\{\theta^i, \theta^*\}\} = \mathbb{1}\{\theta^i \leq \theta^*\}. \quad (32)$$

The logic behind the efficiency of this equilibrium can be summarized as follows. If price discrimination is feasible, the seller can deal with consumers separately. This separation allows the seller to fully appropriate the option value each group of consumers would have if the established product was also supplied competitively. Finally, since the seller appropriates all the option value, she chooses when to sell exactly as if she were maximizing total surplus.
The following figure illustrates the efficiency of the equilibrium by displaying Ann’s value and the profit the seller extracts from her.

![Graph showing seller's profits and equilibrium value for Ann.](image)

**Figure 4.** Seller’s profits and equilibrium value for Ann.

The following figure plots the seller’s total profit as a function of the state.

![Graph showing seller's total profits.](image)

**Figure 5.** Seller’s total profits obtained from Ann and Bob.
5. Equilibrium without price discrimination

In this section, I consider the case in which the seller is constrained to offer Ann and Bob exactly the same price. This assumption is quite natural if we think of Ann and Bob as representations of two different market segments composed of many anonymous consumers, who identify themselves with the mean public opinions $\theta^A$ and $\theta^B$ about the new product, but cannot be individualized by the seller.

Note that the rationale behind the MPE constructed in the previous section suggests that, if the seller cannot engage in price discrimination, she will be forced to concede some rents to inframarginal consumers. As we will see shortly, this generates dynamic inefficiency in the form of over-experimentation. More precisely, I will construct a MPE without price discrimination and show that there are some initial states for which the equilibrium prescribes efficient stochastic paths of product choices, but there are others in which consumers stop using the new product “too late” (for a value of $\theta^i$ lower than $\theta^*$). However, for sufficiently large $t$, consumers choose their purchases efficiently with probability 1.

The claims in the previous paragraph are formalized through four propositions later on this section. However, before proceeding to the analysis leading to these results, it seems convenient to informally discuss the nature of the MPE we are seeking. The seller will serve both Ann and Bob when they are sufficiently pessimistic about the new product (i.e. $\theta^A$ and $\theta^B$ low enough), since in such situation they have fewer incentives to experiment and is cheap to attract them. In this case, the marginal buyer (the one who determines the equilibrium price) will be the consumer with higher $\theta^i$. For example, if $\theta^A > \theta^B$, the marginal buyer would be Ann. Now, suppose we increase the value of $\theta^A$ so that Ann becomes more optimistic. Then, the seller would have to reduce the price in order to keep her buying. At some point, the seller would cease to find the price reduction strategy optimal and, as she raises the price, Ann will switch to the new product and Bob will become the seller’s marginal buyer.
In contrast, if both Ann and Bob are very optimistic about the new product, the seller will not be interested in serving them at all. The reason is that, in order to attract Ann and/or Bob in this circumstance, the seller would have to set very low prices. Instead, she prefers to let Ann and Bob experiment. If they find out that they don’t like the new product so much, the seller will offer them a price low enough to attract them and, at the same time, high enough to be profit maximizing (taking into account what the seller could achieve by letting the most pessimistic consumer experiment a little more). The following figure illustrates this logic anticipating the anatomy of the equilibrium and will be justified by the subsequent analysis.

\[
\begin{align*}
\theta^B & \quad 1 \\
1 & \quad \tilde{\theta}(1) \\
\theta^* & \quad \tilde{\theta}(1) \\
\theta^c & \quad \tilde{\theta}(1) \\
\tilde{\theta}(0) & \quad 0 \\
\end{align*}
\]

**Figure 6.** Experimentation regions and belief dynamics (Ann’s boundary in dashed blue, Bob’s boundary in solid red).
To begin the analysis, note that if $\theta^B = 1$, Bob will never buy from the seller and we are back to the case of the previous section. On the other hand, if $\theta^B = 0$, Bob is a captive client for the seller. The corresponding equilibrium analysis is similar to the case $\theta^B = 1$, except that now the problem of the seller in equation (28) includes the profit obtained from Bob, who can be expected to buy at any price not exceeding $\mu^*$. The equilibrium value for Ann is still given by

$$ V^A(\theta^A, 0) = \frac{\theta^A}{r}. \tag{33} $$

However, the total profit of the seller now satisfies the following HJB equation:

$$ r\Pi(\theta^A, 0) = \max \left\{ 2(\mu^* - \theta^A), \mu^* + \frac{1}{2} V(\theta^A) \frac{d^2 \Pi(\theta^A, 0)}{d\theta^A^2} \right\}. \tag{34} $$

In this equation, $\mu^* - \theta^A$ is the price the seller has to charge to keep Ann as her customer (therefore selling two units of her product), while $\mu^*$ is the price the seller can charge if she decides to forget about Ann, raise the price and concentrate in Bob. Formally, this equation represents the value of an optimal stopping problem of the same kind than the one we solved for the planner in Section 3. The unique solution of the associated free-boundary problem is a cutoff-value pair $(\tilde{\theta}(0), \Pi)$, with the cutoff defined by

$$ \tilde{\theta}(0) := \left( \frac{\alpha \mu^*}{\mu^* - 2\beta} \right) \in (0, \theta^*) \quad \tag{35} $$

and the profits satisfying

$$ r\Pi(\theta^A, 0) = \begin{cases} 
2(\mu^* - \theta^A) & \theta^A \leq \tilde{\theta}(0) \\
\mu^* + (\mu^* - 2\tilde{\theta}(0)) \left( \frac{H(\theta^A)}{H(\tilde{\theta}(0))} \right) & \theta^A > \tilde{\theta}(0).
\end{cases} \tag{36} $$
Note that $\bar{\theta}(0)$ is the maximal value of $\theta^A$ at which the seller is willing to charge Ann’s indifference price (which is lower than Bob’s) in order to sell two units instead of one. The value for Bob is given by

$$rV^B(\theta^A, 0) = \begin{cases} \theta^A & \theta^A \leq \bar{\theta}(0) \\ 0 & \theta^A > \bar{\theta}(0). \end{cases}$$

Note that, when $\theta^A \leq \bar{\theta}(0)$, the seller is targeting Ann, so Bob becomes inframarginal. As a consequence, he gets the positive rent $V^B(\theta^A, 0) - 0/r > 0$ in excess of the expected value of unconditional continuation. On the other hand, Ann never becomes inframarginal in the locus $\theta^B = 0$ and so she gets no such a rent. We can see this by noting that $V^A(\theta^A, 0) - \theta^A/r = 0$.

The analysis when $\theta^B$ is similar. For $\theta^A \geq \theta^B$ the seller will consider whether is convenient to attract Ann by setting the price at $\hat{p}(\theta^A, \theta^B) = \mu^* - \theta^A$ or sell only to Bob. Note that $\mu^* - \theta^A$ is the price which renders Ann indifferent between buying from the seller and experimenting with the new product. If the seller sets the price at $\hat{p}^A(\theta^A, \theta^B)$, Ann’s value will satisfy

$$rV^A(\theta^A, \theta^B) = \theta^A + \frac{1}{2} v(\theta^A) \frac{\partial^2 V^A(\theta^A, \theta^B)}{\partial \theta^A^2} \quad \theta^A \in [\bar{\theta}(\theta^B), 1],$$

where $\bar{\theta}(\theta^B)$ be the maximal value of $\theta^A$ at which the seller wants to sell to both consumers when Bob’s belief is $\theta^B \leq \theta^A$. Note that $rV^A(\theta^A, \theta^B) = \theta^A$ for $\theta^A \in [\theta^B, \bar{\theta}(\theta^B)]$. The unique solution to this initial value problem is given by

$$rV^A(\theta^A, \theta^B) = \theta^A.$$  

Note that we have not yet determined the equilibrium cutoff $\bar{\theta}(\theta^A)$. Before doing that, we need to complete the analysis of equilibrium prices for low $\theta^B$. So what is the seller’s pricing policy for $\theta^A \in (\bar{\theta}(\theta^B), 1]$? Since the seller no longer wants to serve Ann, she goes after Bob’s segment and offers him his indifference price $\hat{p}^B(\theta^A, \theta^B) = \mu^* - \theta^B$. 

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As a consequence, for $\theta^A > \tilde{\theta}(\theta^B)$, we have

$$rV^B(\theta^A, \theta^B) = \mu^* - \bar{p}^B(\theta^A, \theta^B) + \frac{1}{2} v(\theta^A) \frac{\partial^2 V^B(\theta^A, \theta^B)}{\partial \theta^A^2} \quad (40)$$

Then, Bob’s value satisfies the ODE

$$rV^B(\theta^A, \theta^B) = \theta^B + \frac{1}{2} v(\theta^A) \frac{\partial^2 V^B(\theta^A, \theta^B)}{\partial \theta^A^2}.$$ \quad (41)

Since $V^B$ should be continuous, we can solve (41) subject to the initial condition

$$rV^B(\tilde{\theta}(\theta^B), \theta^B) = \mu^* - \bar{p}(\tilde{\theta}(\theta^B), \theta^B) = \tilde{\theta}(\theta^B), \quad (42)$$

The solution is

$$rV^B(\theta^A, \theta^B) = \begin{cases} 
\theta^B + (\tilde{\theta}(\theta^B) - \theta^B) \left( \frac{H(\theta^A)}{H(\tilde{\theta}(\theta^B))} \right) & \theta^B \leq \tilde{\theta}(\theta^B) < \theta^A \\
\theta^A & \theta^B \leq \theta^A \leq \tilde{\theta}(\theta^B). \end{cases} \quad (43)$$

We now need to determine the optimal cutoff for the seller. Her profits are determined by the HJB equation:

$$r\Pi(\theta^A, \theta^B) = \max \left\{ 2\bar{p}^A(\theta^A, \theta^B), \bar{p}^B(\theta^A, \theta^B) + \frac{1}{2} v(\theta^A) \frac{d^2 \Pi(\theta^A, \theta^B)}{d \theta^A^2} \right\}$$

$$= \max \left\{ 2(\mu^* - \theta^A), \mu^* - \theta^B + \frac{1}{2} v(\theta^A) \frac{d^2 \Pi(\theta^A, \theta^B)}{d \theta^A^2} \right\} \quad (44)$$

The associated free-boundary problem is solved by

$$r\Pi(\theta^A, \theta^B) = \begin{cases} 
2(\mu^* - \theta^A) & \theta^A \leq \tilde{\theta}(\theta^B) \\
\mu^* - \theta^B + (\mu^* - \tilde{\theta}(\theta^B) + \theta^B - \tilde{\theta}(\theta^B)) \left( \frac{H(\theta^A)}{H(\tilde{\theta}(\theta^B))} \right) & \theta^A > \tilde{\theta}(\theta^B) \end{cases} \quad (45)$$
with the cutoff $\bar{\theta}(\theta^B)$ defined for low $\theta^B$ by

$$
\bar{\theta}(\theta^B) = \frac{\alpha(\mu^* + \theta^B)}{\mu^* + \theta^B - 2\beta}.
$$

(46)

Note that $\bar{\theta}(\theta^B)$ is increasing in $\theta^B$ and that $\bar{\theta}(1) \in (0,1)$. Using this cutoff, we can give a precise meaning to what we meant by “low $\theta^B$”. That is, the previous analysis is valid in the region $\bar{\theta}(\theta^B) \leq \theta^B$. This region can be easily verified to have the form $[0, \theta^c]$, where the critical point $\theta^c$ is given by

$$
\theta^c := \frac{1}{2}(1 + \beta - \mu^* - \sqrt{4\alpha\mu^* + (1 + \beta - \mu^*)^2}) \in (0, \theta^*)
$$

(47)

Up to now, we have constructed $\hat{p}$, $V^A$, $V^B$ and $\Pi$ in the strips $[0,1] \times [0, \theta^c]$ and $[0,1] \times \{1\}$. The construction for the strips $[0, \theta^c] \times [0,1]$ and $\{1\} \times [0,1]$ is symmetric. It remains to describe the equilibrium in the box $[\theta^c, 1] \times [\theta^c, 1]$.

Along the diagonal $\{(\theta, \theta) | \theta \in [\theta^c, \theta^*]\}$, the equilibrium price is $\hat{p}(\theta, \theta) = \mu^* - \theta$ and it is optimal to serve both market segments. It follows that we can extend the definition of $\bar{\theta}$ in equation (46) beyond $\theta^c$:

$$
\bar{\theta}(\theta^B) := \begin{cases} 
\frac{\alpha(\mu^* + \theta^B)}{\mu^* + \theta^B - 2\beta} & \theta^B \in [0, \theta^c] \\
\theta^B & \theta^B \in [\theta^c, \theta^*].
\end{cases}
$$

(48)

With this extended definition, the value functions for Ann and Bob are again given by equations (39) and (43), respectively. As an instance of equation (45), profits are given in this region by:

$$
r\Pi(\theta^A, \theta^B) = \begin{cases} 
2(\mu^* - \theta^A) & \theta^A = \theta^B \\
(\mu^* - \theta^B)\left(1 + \frac{H(\theta^A)}{H(\theta^B)}\right) & \theta^A > \theta^B
\end{cases}
$$

(49)

I shall finish the equilibrium construction by describing behavior in the upper box $(\theta^*, 1) \times (\theta^*, 1)$, where consumers are most optimistic about the new product.
It is intuitive that the equilibrium should prescribe experimentation until they escape the region, since not even the planner would suggest them to consume the seller’s product. The seller can induce this behavior by setting

\[ \hat{\rho}(\theta^A, \theta^B) \geq \mu^* - \theta^* \quad (\theta^A, \theta^B) \in (\theta^*, 1) \times (\theta^*, 1). \tag{50} \]

Note that we know the value functions in all the four boundaries of \((\theta^*, 1) \times (\theta^*, 1)\). To describe the value functions, define the first-exit time of \((\theta^*, 1) \times (\theta^*, 1)\) as

\[ \tau^* := \inf\{t > 0 | (\theta^A_t, \theta^B_t) \notin (\theta^*, 1) \times (\theta^*, 1)\}. \tag{51} \]

The value function of Ann satisfies

\[ V^A(\theta^A, \theta^B) = \mathbb{E}\left\{ \int_0^{\tau^*} e^{-rt} \theta^A_t \, dt + e^{-r\tau^*} V^A(\theta^A_{\tau^*}, \theta^B_{\tau^*}) \middle| (\theta^A_0, \theta^B_0) = (\theta^A, \theta^B) \right\}. \tag{52} \]

The value function for Bob satisfies \( V^B(\theta^A, \theta^B) = V^A(\theta^B, \theta^A) \). The profit function of the seller can be represented through the following expectation:

\[ \Pi(\theta^A, \theta^B) = \mathbb{E}\{e^{-r\tau^*} \Pi(\theta^A_{\tau^*}, \theta^B_{\tau^*}) | (\theta^A_0, \theta^B_0) = (\theta^A, \theta^B)\}. \tag{53} \]

Product choices in this region are fully efficient, so

\[ V^A(\theta^A, \theta^B) = \frac{\theta^A}{r} \quad V^B(\theta^A, \theta^B) = \frac{\theta^B}{r} \quad \Pi(\theta^A, \theta^B) = S(\theta^A, \theta^B) - \frac{\theta^A + \theta^B}{r}. \tag{54} \]

It follows that the switching cutoff for Ann is finally extended to:

\[ \tilde{\bar{\theta}}(\theta^B) := \begin{cases} \frac{\alpha(\mu^* + \theta^B)}{\mu^* + \theta^B - 2\beta} & \theta^B \in [0, \theta^c] \\ \frac{\theta^B}{\theta^*} & \theta^B \in [\theta^c, \theta^*] \\ \theta^* & \theta^B \in [\theta^*, 1]. \end{cases} \tag{55} \]

The cutoff for Bob is defined symmetrically.
<table>
<thead>
<tr>
<th>Region</th>
<th>Conditions</th>
<th>Ann’s choice</th>
<th>Bob’s choice</th>
<th>Marginal consumer</th>
<th>$rV^A(\theta^A, \theta^B)$</th>
<th>$rV^B(\theta^A, \theta^B)$</th>
<th>$r\Pi(\theta^A, \theta^B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>$0 \leq \theta^A = \theta = \theta^B \leq \theta^c$</td>
<td>seller</td>
<td>seller</td>
<td>Ann &amp; Bob</td>
<td>$\theta$</td>
<td>$\theta$</td>
<td>$2(\mu^* - \theta)$</td>
</tr>
<tr>
<td>$R_2$</td>
<td>$0 \leq \theta^B &lt; \theta^A \leq \tilde{\theta}(\theta^B) &lt; \theta^c$</td>
<td>seller</td>
<td>seller</td>
<td>Ann</td>
<td>$\theta^A$</td>
<td>$\theta^A$</td>
<td>$2(\mu^* - \theta^A)$</td>
</tr>
<tr>
<td>$R_2'$</td>
<td>$0 \leq \theta^A &lt; \theta^B \leq \tilde{\theta}(\theta^A) &lt; \theta^c$</td>
<td>seller</td>
<td>seller</td>
<td>Bob</td>
<td>$\theta^B$</td>
<td>$\theta^B$</td>
<td>$2(\mu^* - \theta^B)$</td>
</tr>
<tr>
<td>$R_3$</td>
<td>$0 \leq \theta^B \leq \theta^c$, $\tilde{\theta}(\theta^B) &lt; \theta^A$</td>
<td>new</td>
<td>seller</td>
<td>Bob</td>
<td>$\theta^A$</td>
<td>$\theta^A + \frac{(\tilde{\theta}(\theta^B) - \theta^B)H(\theta^A)}{H(\tilde{\theta}(\theta^B))} + \frac{(\mu^* + \theta^B - 2\tilde{\theta}(\theta^B))H(\theta^A)}{H(\tilde{\theta}(\theta^B))}$</td>
<td>$\theta^B$</td>
</tr>
<tr>
<td>$R_3'$</td>
<td>$0 \leq \theta^A \leq \theta^c$, $\tilde{\theta}(\theta^A) &lt; \theta^B$</td>
<td>seller</td>
<td>new</td>
<td>Ann</td>
<td>$\theta^A + \frac{(\tilde{\theta}(\theta^A) - \theta^A)H(\theta^B)}{H(\tilde{\theta}(\theta^A))}$</td>
<td>$\theta^B$</td>
<td>$\mu^* - \theta^A + \frac{(\mu^* + \theta^A - 2\tilde{\theta}(\theta^A))H(\theta^B)}{H(\tilde{\theta}(\theta^A))}$</td>
</tr>
<tr>
<td>$R_4$</td>
<td>$\theta^<em>, \theta^</em> &lt; \theta^B$</td>
<td>new</td>
<td>new</td>
<td>-</td>
<td>$\theta^A$</td>
<td>$\theta^B$</td>
<td>$r\Pi(\theta^A, \theta^B)$</td>
</tr>
</tbody>
</table>

**Table 1.** Equilibrium regions, product choices, marginal consumers and value functions.
The following is the main result of this section:

**Proposition 3.** The Markov strategy profile \((\hat{p}, \hat{p}^A, \hat{p}^B)\) is a MPE.

Comparing with the socially efficient dynamic product choices, this MPE features over-experimentation in some regions of the state space. The following result describes the efficiency properties of this equilibrium and shows exactly when it induces dynamically inefficient outcomes.

**Proposition 4.** Consider the (random) path of product choices induced by \((\hat{p}, \hat{p}^A, \hat{p}^B)\) for different prior beliefs \(\theta_0 = (\theta_0^A, \theta_0^B)\). The outcome is inefficient with positive probability if and only if

\[
(0 \leq \theta_0^A < \theta^* \land \bar{\theta}(\theta_0^A) < \theta_0^B < 1) \lor (0 \leq \theta_0^B < \theta^* \land \bar{\theta}(\theta_0^B) < \theta_0^A < 1). \tag{56}
\]

Condition (56) corresponds to a region \(I := (\text{int} \ R_3 \setminus \{1\} \times [0,1]) \cup (\text{int} \ R_3' \setminus [0,1] \times \{1\})\).

In intuitive terms, if both players start with a symmetric enough prior, the outcome is efficient because the seller’s incentives to sell to both consumers are relatively similar. Note that Proposition 4 implies

\[
\Pi(\theta) + V^A(\theta) + V^B(\theta) = S(\theta) \tag{57}
\]

whenever condition (56) is violated. In contrast, if the prior is sufficiently asymmetric, the outcome might be inefficient because the seller may prefer to bet on the event in which the more optimistic consumer experiences a bad history by waiting to reduce the price beyond what is required by efficiency.
The stochastic dynamics implied by this MPE is depicted in Figure 7 below.

![Figure 7. Dynamics: inefficient product choice (textured area) and rest points (thick black).](image)

The arrows in Figure 7 represent the directions in which the state can move. The thick black lines and black dots correspond to absorbing states where beliefs don’t change and the stochastic dynamics is at rest. Note that the equilibrium implements the efficient action in all these absorbing states. Moreover, beliefs are martingales under the equilibrium strategies and therefore converge.
The next result exploits these two observations to show that inefficiency is a transient phenomenon.

**Proposition 5.** For every \( \theta_0 \), the MPE \((\hat{p}, \tilde{p}^A, \tilde{p}^B)\) prescribes an efficient product choice in finite time with probability 1.

This means that almost every path induced by \((\hat{p}, \tilde{p}^A, \tilde{p}^B)\) eventually leads to the planner’s allocation independently of the prior. For example, when \( \theta_0 \in R_3 \), we have

\[
\Pr \left\{ \lim_{t \to \infty} \theta^A_t = \bar{\theta}(\theta^B) \left| \theta^A_0 \in \left[ \bar{\theta}(\theta^B_0), 1 \right) \right. \right\} = \frac{1 - \theta^A_0}{1 - \bar{\theta}(\theta^B)} \quad (58)
\]

and

\[
\Pr \left\{ \lim_{t \to \infty} \theta^A_t = 1 \left| \theta^A_0 \in \left[ \bar{\theta}(\theta^B_0), 1 \right) \right. \right\} = \frac{\theta^A_0 - \bar{\theta}(\theta^B)}{1 - \bar{\theta}(\theta^B)} \quad (59)
\]

Given this result, it is natural to wonder how much inefficiency can actually take place. One way of answering this question is to measure the size of region of the state space in which equilibrium product choices are not efficient. So, let \( L(\mu^*, r, \sigma) \) denote the Lebesgue measure of the textured area in Figure 7. Then, we have

**Proposition 6.** The size of the inefficient area associated with the MPE \((\hat{p}, \tilde{p}^A, \tilde{p}^B)\) is

\[
L(\mu^*, r, \sigma) = (\theta^*)^2 + (\theta^c)^2 - 4\alpha \beta \ln \left( 1 + \frac{\theta^c}{\mu^* - 2\beta} \right) - 2\alpha \theta^c. \quad (60)
\]

Moreover,

\[
\lim_{\sigma \to \infty} L(\mu^*, r, \sigma) = \lim_{r \to \infty} L(\mu^*, r, \sigma) = \frac{(\mu^*)^2}{2}. \quad (61)
\]
It follows that $L(\mu^*, r, \sigma)$ can get arbitrarily close to 1/2, which means that almost half of the state space can induce an initial inefficient equilibrium product choice when the signal noise and/or the discount rate are sufficiently high. Moreover, if we let $\sigma \to \infty$ belief dynamics becomes arbitrarily slow. This means that there is no bound on the persistence of the inefficient product choices.

The following figure plots the difference between the maximal total surplus and the total surplus obtained in equilibrium:

![Figure 8. Loss due to inefficiency across states.](image)

As illustrated by Figure 7, dynamic inefficiency may only arise if the prior lies in $I \subset R_3 \cup R'$. Conversely, if the prior lies in $[0,1]^2 \setminus (R_1 \cup R_2 \cup R'_2 \cup \overline{R}_4$, consumers will choose products just as the planner with probability 1. This means that in those regions, the equilibrium achieves maximal total surplus, as shown in Figure 8.
A noteworthy feature of this MPE is that prices are discontinuous along the boundaries $\theta^A = \bar{\theta}(\theta^B)$ and $\theta^B = \bar{\theta}(\theta^A)$. For $\theta^A, \theta^B < \theta^c$, this discontinuity arises from the fact that, at the switching boundaries, the seller raises the price to focus on her new marginal consumer, which is less optimistic about the new product than the consumer who starts experimenting. The following figure shows the price jumps along the cutoffs described in (55) for both Ann and Bob:

![Figure 9](image)

**Figure 9.** The equilibrium price is discontinuous at switching boundaries.

Note that, for fixed $\theta^B < \theta^c$, the equilibrium price is locally non-increasing in $\theta^A$ at every point of continuity. However, due to the jump at $\bar{\theta}(\theta^B)$, the price is not monotonic in a global sense. Intuitively, prices are lower when consumers have sufficiently symmetric beliefs and the seller wants to attract them both.
The following figure illustrates the lack of monotonicity of $\hat{p}(\theta^A, \theta^B)$ fixing the belief of Bob at $\theta^B \in (\theta^c, \theta^*)$:

![Figure 10. The equilibrium price is non-monotonic.](image)

Note that, while prices are discontinuous, the value functions of all players are continuous functions of the state. However, they are not differentiable in all regions. For instance, the seller’s profit function has a kink along the diagonal if $\theta^A = \theta^B \in [0, \theta^*)$. The reason is that, if beliefs start at $\theta^A_0 > \theta^B_0 \in (\theta^c, \theta^*)$ and $\theta^A$ decreases, Bob drops out of the market just after crossing $\theta^A = \theta^B$. At that point, further reductions in $\theta^A$ increase profits at a higher rate because Ann becomes the only consumer in the market.
The following contour plot illustrates this phenomenon:

![Contour Plot](image)

**Figure 11.** Equilibrium isoprofit lines.

Similarly, Ann’s value function has kinks in the boundary between $R'_3$ and $R'_2$, when her customer status passively changes from marginal to inframarginal. The reason is that she suddenly gets a rent without switching her actions, as the seller’s pricing strategy starts targeting Bob who is more optimistic about the new product and therefore requires a lower price in order to buy.
How much can the seller gain from price discrimination? Combining the analysis of the previous section with that of this one, we can plot the difference between the profit with and without price discrimination over the state space.

![Figure 12. Gains from price discrimination.](image)

On one hand, as we can see in Figure 12, the most significant gains occur when expected valuations are moderately asymmetric. On the other hand, there is nothing to gain from price discrimination if there is no asymmetry, if the asymmetry is extreme or if full experimentation is efficient.
6. Extensions

This section briefly explores some natural extensions of the basic model: allowing commitment, more market segments, asymmetries, dominated new product, more possiblevaluations, strategic pricing of the new product instead of the known one and positive switching costs.

6.1 Commitment

The equilibrium analysis of the basic model was performed under the assumption that the seller cannot commit. However, commitment is not important for the anatomy of the equilibrium and allowing the seller to commit to a price path at $t = 0$ does not change the outcome.

In the case in which price discrimination is feasible, it is clear that the seller would find optimal to commit to the MPE strategy described in Section 4. The reason is that, in this equilibrium, consumers obtain exactly the value of their outside options at every state. Thus, the seller’s payoff is the best payoff she can obtain through any mechanism in which consumers participate voluntarily. This would be true even if individual consumers could affect the aggregate amount of experimentation in their segment.

If price discrimination is not feasible, the seller will be forced to give some rents to inframarginal consumers, but commitment will not mitigate this loss. Since individual consumers are “informationally small”, they behave as if they were myopic. Thus committing to price in a particular way in the future does not allow the seller to affect consumer’s present choices.
6.2 More market segments

It seems natural to wonder whether one can extend the basic model presented in the previous sections to the case of more than two market segments. On one hand, the equilibrium analysis with price discrimination allows any number of segments without any essential modification. On the other hand, if price discrimination is not feasible, constructing a MPE becomes more cumbersome. However, I believe it should still be possible to construct a monotonic equilibrium by ordering beliefs and letting the seller switch at optimal cutoffs. To be more specific, suppose $N \geq 3$ and let the state $\theta$ satisfy $\theta^1 < \cdots < \theta^N$. At that state, there will be $n \in \{1, \ldots, N\}$ such that consumers $\{1, \ldots, n\}$ are buying from the seller in a neighborhood of $\theta$, but consumers $\{|i| i > n\}$ are not. If $n < N$ and $\bar{p}^i$ denotes the indifference price for consumer $i \in \{1, \ldots, N\}$, the seller’s equilibrium profits will solve the following HJB equation:

$$r\Pi(\theta) = \max \left\{ (n + 1)\bar{p}^{n+1}, n\bar{p}^n + \frac{1}{2} \nu(\theta^{n+1}) \frac{\partial^2 \Pi(\theta)}{\partial \theta^{n+1^2}} \right\} + \sum_{i=n+2}^{N} \frac{1}{2} \nu(\theta^i) \frac{\partial^2 \Pi(\theta)}{\partial \theta^{i^2}}. \quad (62)$$

Solving for the equilibrium is now more complex because value functions are determined by partial differential equations. However, we can still get some intuition about the efficiency properties of the equilibrium analyzing what happens with the equilibrium cutoffs in some regions of the state space. For example, consider the extreme set of states in which $\theta^1 = \theta^2 = \cdots = \theta^{N-1} = 0 < \theta^N$. In that region, the seller will face the problem

$$r\Pi(\theta) = \max \left\{ N\bar{p}^{N}, (N - 1)\mu^* + \frac{1}{2} \nu(\theta^{N+1}) \frac{\partial^2 \Pi(\theta)}{\partial \theta^{N^2}} \right\} \quad (63)$$

in any MPE.
The solution is

\[ r\Pi(\theta) = \begin{cases} 
N(\mu^* - \theta^N) & \theta^N \leq \bar{\theta}^N \\
(N - 1)\mu^* + (\mu^* - N\bar{\theta}^N) \frac{H(\theta^N)}{H(\bar{\theta}^N)} & \theta^N > \bar{\theta}^N, 
\end{cases} \quad (64) \]

where the cutoff \( \bar{\theta}^N \) is given by

\[ \bar{\theta}^N := \frac{\alpha \mu^*}{\mu^* + (\alpha - 1)N} \quad (65) \]

It follows that

\[ \lim_{N \to \infty} \bar{\theta}^N = 0. \quad (66) \]

The interpretation is very simple. As the seller has more captive consumers, she has less incentives to attract consumer \( N \). This implies that the equilibrium cutoff when all the other \( N - 1 \) consumers are captive deviates more and more from efficiency and, in the limit, converges to zero.
6.3 Asymmetric consumers

It is also important to consider what happens when consumers are asymmetric. Allowing heterogeneity in $\mu^*$, $\sigma$ and different segment sizes seem the most interesting form of asymmetries to analyze.

Again, the equilibrium analysis with price discrimination goes through, although the efficient cutoffs for Ann and Bob will be determined by their idiosyncratic parameters $\mu^*$ and $\sigma$. Due to the full separability of the problem, having different segment sizes does not change the equilibrium nor the solution of the planner’s problem.

Without price discrimination, allowing for different valuations for the established product shifts the locus in which the identity of the buyer who is most willing to pay for the seller’s product switches. The switching boundaries $\bar{\theta}^A(\theta)$ and $\bar{\theta}^B(\theta)$ will also be different. As a result, the equilibrium pricing function changes. Similarly, allowing different market segment sizes changes the switching boundaries and the pricing function, but not the efficient allocation.
6.4 Dominated new product

Here I briefly analyze the case $\mu^* \geq 1$. This is the right assumption if we are modeling a situation in which the new product is at most as good as the original and the key difference is that consumers can get it for free (or at a lower price). One example of this situation is original software versus pirated copies (which may work just as the original or malfunction at some point).

In this case, it is always efficient to have consumers using the seller’s product. As a consequence, we have $\theta^* = 1$ and the MPE with price discrimination is given by $\hat{p}^i(\theta) = \bar{p}^i(\theta) = \mu^* - \theta^i$.

Without price discrimination, we will still have $\hat{p}(\theta^B) = \frac{\alpha(\mu^* + \theta^B)}{\mu^* + \theta^B - 2\beta}$ (67)

For $\mu^* = 1$, we find $\theta^c = 1$ so the switching boundaries of Ann and Bob meet at $(\theta_A, \theta_B) = (1,1)$. For $\mu^* \in (1,2)$, the boundaries do not meet but $\bar{\theta}(0) < 1$, so there are partial experimentation regions in which only one segment buys the new product. Of course, in both cases, the region $R_4$ no longer exists. For $\mu^* \geq 2$, the seller’s product is too good and consumers will prefer to pay its price in every state.

Interestingly, the asymptotic efficiency result in Proposition 5 does not hold for $\mu^* \in [1,2)$. The reason is that if, for example, $\mu^A = 1$, $\theta^B_0 < \theta^* \text{ and } \theta^A_0 > \bar{\theta}(\theta^B_0)$, there is positive probability that $\theta_t \rightarrow (\theta^B_0,1)$. But, in this event, $\theta_t$ remains trapped in the inefficient portion of $R_3$ forever.
6.5 Continuum of valuations

The analysis presented in the previous sections was restricted to binary valuations. In applications, it is sometimes important to allow for many or even a continuum of possible valuations. For example, there might be no a priori bound on the valuation for the new product. In this subsection, I explore an extension in this direction by focusing on the case in which \( \mu \in \mathbb{R} \) and prior beliefs are Gaussian.

In this case, the beliefs dynamics becomes non-stationary. This is because experimentation reduces belief dispersion over time independently of what happens with the posterior mean (something impossible if the prior has a two-point support). Consider the following Gaussian belief parametrization for Ann

\[
\mu^A|F_t^A \sim N(m_t^A, v_t^A),
\]

where \( m_t^A \) is her posterior mean and \( v_t^A \) her posterior variance. More specifically, we can define the conditional moments

\[
m_t^A := \mathbb{E}[\mu_t^A|F_t^A] = \frac{m_0^A + v_0^A X_t^A}{1 + v_0^A t}, \quad v_t^A := \mathbb{V}[\mu_t^A|F_t^A] = \frac{v_0^A}{1 + v_0^A t}.
\]

Then, her individual state can be described by the pair \((m_t^A, v_t^A)\) which follows

\[
dm_t^A = (1 - q_t^A)v_t^A d\tilde{Z}_t^A, \quad dv_t^A = -(1 - q_t^A)dt,
\]

where \( \{\tilde{Z}_t^A\} \) is Ann’s innovation process defined as before and \( q_t^A \) is her purchasing strategy. Note that \( \{m_t^A\} \) is a martingale and \( \{v_t^A\} \) is non-increasing (and, conditional on \( q_t^A = 1 \), decreases deterministically).
The planner problem is still separable but more complex. It can be shown that the maximal surplus from Ann $S^A(m^A, v^A)$ is a non-decreasing convex function which equals $\mu^*/r$ below a threshold $s(v^A)$ and is increasing above $s(v^A)$. The following figure illustrates the shape of $S^A$ for fixed $v^A$:

![Figure 13. Maximal surplus from Ann for fixed $v^A$.](image)

I will now sketch a method to find $s(v^A)$. If the function $S^A$ is smooth enough, it will satisfy the following HJB equation

$$
 r S^A(m^A, v^A) = \max \left\{ \mu^*, m^A + (v^A)^2 \left( \frac{1}{2} \frac{\partial^2 S^A(m^A, v^A)}{\partial m^A^2} - \frac{\partial S^A(m^A, v^A)}{\partial v^A} \right) \right\},
$$

By extending the verification argument in the proof of Proposition 1, any solution to this equation with a $C^1$ free-boundary can be shown to be the maximal total surplus the planner can obtain through Ann.
It is possible to reduce the HJB equation (71) to an homogeneous heat equation and adapt the arguments in Kolodner (1956) to prove that any solution corresponding to a $C^1$ boundary will be of the form

$$
 rS^A(m^A, v^A) = \begin{cases} 
 m^A + \frac{1}{2} \int_0^{v^A} f(m^A, v^A, u, s) du & m^A > s(v^A) \\
 \mu^* & m^A \leq s(v^A), 
\end{cases} 
$$

(72)

where $f$ is defined by

$$
 f(m^A, v^A, u, s) := \left( \frac{e^{-r\left(\frac{1}{u} - \frac{1}{v}\right)}}{\sqrt{v-u}} \right) \phi \left( \frac{m^A - s(u)}{\sqrt{v-u}} \right) \left[ 1 - \left( \frac{m^A - s(u)}{v-u} \right) - 2s'(u) \right] s(u) 
$$

(73)

and $s: [0, \infty) \to \mathbb{R}$ represents the free-boundary. It turns out that $s$ is characterized by the following functional equation

$$
 s(v^A) = \mu^* - \int_0^{v^A} f(s(v^A), v^A, u, s) du 
$$

(74)

Equation (74) is a non-linear Volterra integro-differential equation of the 2nd kind. While the question of existence of a solution is left for future research, if such solution exists, it must be unique and provides the planner’s efficient threshold.

Assuming that we can solve (74), the function $S^A$ will be smooth enough for the HJB equation (71) to be valid. Then, it would be possible to construct an efficient MPE with price discrimination, just as we did in the binary case. The equilibrium analysis without price discrimination can also be pursued along similar lines, but it will be technically more challenging. For instance, profit maximization will involve a higher dimensional switching boundary (for example, if $m^B$ is sufficiently low, the seller will serve the whole market when $m^A \leq \bar{m}^A(v^A, m^B, v^B)$).
6.6 Strategic pricing of the new product

Now suppose that the seller controls the price of the new product and tries to penetrate an established competitive market. This assumption fits the situation arising after the seller innovates and obtains a patent and corresponds to an opposite location of market power relative to the basic model.

It terms of efficiency, it doesn’t matter whether the seller is pushing the new product or defending her previous monopoly. The allocation problem of the planner is exactly the same as that of the previous case. Hence, Proposition 1 applies.

The HJB equations for MPE are now

\[
 r\Pi = \sup_{p \geq 0} \left\{ 1\{p^A \leq \bar{p}^A\} \left( p^A + \frac{\nu(\theta^A)}{2} \frac{\partial^2 \Pi}{\partial \theta^2} \right) + 1\{p^B \leq \bar{p}^B\} \left( p^B + \frac{\nu(\theta^B)}{2} \frac{\partial^2 \Pi}{\partial \theta^B} \right) \right\} \tag{75}
\]

\[
 rV^A = \max\{\mu^*, \theta^A - \hat{p}^A(\theta)\} + \frac{1}{2} \nu(\theta^A) \frac{\partial^2 V^A}{\partial \theta^2} \{\hat{p}^A > \bar{p}^A\} + \frac{1}{2} \nu(\theta^B) \frac{\partial^2 V^A}{\partial \theta^B^2} \{\hat{p}^B > \bar{p}^B\} \tag{76}
\]

\[
 rV^B = \max\{\mu^*, \theta^B - \hat{p}^B(\theta)\} + \frac{1}{2} \nu(\theta^B) \frac{\partial^2 V^B}{\partial \theta^2^2} \{\hat{p}^B > \bar{p}^B\} + \frac{1}{2} \nu(\theta^A) \frac{\partial^2 V^B}{\partial \theta^A^2} \{\hat{p}^A > \bar{p}^A\}. \tag{77}
\]

We can construct a MPE using these equations along the lines of the previous analysis. If price discrimination is feasible, the MPE will be efficient. The price charged to Ann when \( \hat{p}^A(\theta) \leq \bar{p}^A(\theta) \) will be \( \hat{p}^A(\theta) = \bar{p}^A(\theta) = \theta^A - \mu^* \). Hence, assuming that \( V^A(\theta) \) does not depend on \( \theta^B \), we have \( V^A(\theta) = \mu^*/r \). The equilibrium price for Ann can therefore be set at

\[
 \hat{p}^A(\theta) = \max\{\theta^A, \theta^*\} - \mu^* \tag{78}
\]

with which the seller will obtain the following profit from Ann:

\[
 r\Pi^A(\theta^A) = \begin{cases} 
 \theta^A - \mu^* + (\mu^* - \theta^*) \left( \frac{H(\theta^A)}{H(\theta^*)} \right) & \theta^A > \theta^* \\
 0 & \theta^A \leq \theta^*.
\end{cases} \tag{79}
\]
Without price discrimination, the MPE will feature under-experimentation in some states. The reason is that, since the seller can target her more optimistic customers, she does not want to sell the new product at a price low enough to achieve the efficient level of market penetration. We can check this claim by computing $\tilde{\theta}^A(1)$, the maximal value of $\theta^A$ such that Ann buys the established product when $\theta^B = 1$. Note that Ann is the marginal consumer for the seller. Thus, she is priced to indifference in equilibrium and her value function becomes $V^A(\theta^A, 1) = \mu^*/r$ as the seller appropriates all her option value. The equilibrium price will be

$$\hat{p}(\theta^A, 1) = \begin{cases} \theta^A - \mu^* & \theta^A > \tilde{\theta}^A(1) \\ 1 - \mu^* & \theta^A \leq \tilde{\theta}^A(1). \end{cases}$$

(80)

The problem of the seller is now

$$r\Pi(\theta^A, 1) = \max \left\{ 1 - \mu^*, 2(\theta^A - \mu^*) + \frac{1}{2} v(\theta^A) \frac{\partial^2 \Pi(\theta^A, 1)}{\partial \theta^A} \right\}.$$  

(81)

The solution satisfies:

$$r\Pi(\theta^A, 1) = \begin{cases} 2(\theta^A - \mu^*) + [1 + \mu^* - 2\tilde{\theta}^A(1)] \left( \frac{H(\tilde{\theta}^A)}{H(\tilde{\theta}^A(1))} \right) & \theta^A > \tilde{\theta}^A \\ 1 & \theta^A \leq \tilde{\theta}^A, \end{cases}$$

(82)

where

$$\tilde{\theta}^A(1) = \frac{\mu^* + 1}{2} + \left( \frac{H(\tilde{\theta}^A)}{H'(\tilde{\theta}^A)} \right).$$

(83)

It follows that $\tilde{\theta}^A(1) > \theta^*$ and Ann stops consuming the new product too soon.
6.7 Positive switching costs

If switching between products is costly, the nature of the efficient allocation will change. The reason is that the state space itself must grow to include current product choices. In this way, Ann will have a switching cutoff $\theta_{new}^*$ when she is consuming the new product and a different (higher) switching cutoff $\theta_{old}^*$ when she is consuming the old product. The gap between $\theta_{new}^*$ and $\theta_{old}^*$ is due to the fact that, since switching is costly, Ann will wait until the expected benefit of switching compensates the cost.

The maximal surplus for Ann when experimenting still satisfies

$$rS_{new}^A = \theta^A + \frac{1}{2} \nu(\theta^A) \left( \frac{d^2 S_{new}^A}{d\theta^A} \right) > \theta_{new}^*.$$  \hspace{1cm} (84)

However, now we will have $S_{new}^A(\theta_{new}^*) + k = \mu^*$, where $k > 0$ is the switching cost. By inspection, we realize that having $k > 0$ is equivalent to a decrease in $\mu^*$. Hence, it follows from Proposition 1 that the efficient cutoff is

$$\theta_{new}^* := \frac{\alpha(\mu^* - k)}{(\mu^* - k) - \beta}.$$  \hspace{1cm} (85)

Note that

$$\lim_{k \to \infty} \theta_{new}^* = \alpha < 0$$

This is natural, since, if the cost of switching is too large, Ann will never want to stop experimenting. On the other hand, the maximal surplus for Ann when consuming the old product satisfies $S_{old}^A(\theta_{old}^*) + k = S_{new}^A(\theta_{new}^*)$. It follows that

$$\theta_{old}^* := \frac{\alpha(\mu^* + k)}{(\mu^* + k) - \beta} > \theta_{new}^*.$$  \hspace{1cm} (86)

We thus see our previous claim confirmed. Of course, if Ann were consuming the old product at $\theta^A > \theta_{old}^*$, efficiency would require an immediate switch.
7. Final remarks

The analysis in this paper sheds light on the dynamic pricing problem of a monopolistic seller who sees her dominant position challenged by a competitive experience substitute. I used a simple model in continuous time to study the dynamic efficiency effects of price discrimination across different market segments. I constructed MPE and showed that, when the seller can charge different prices, she chooses to maximize total surplus in equilibrium. In contrast, when the seller is constrained to charge the same price to all consumers, they may experiment too much with the new product relative to the efficient allocation. This dynamic inefficiency can be very persistent, especially if learning is slow. However, the inefficiency turns out to be transient, since I show that equilibrium strategies end up prescribing an efficient product choice in finite time with probability 1.

Although the MPE constructed seem natural, the question of uniqueness is currently unresolved. Clearly multiple equilibrium prices are possible in the states in which no consumer buys the seller’s product. This multiplicity seems harmless, since it does not affect payoffs. I believe that, leaving this multiplicity aside, the equilibrium is unique. However, this assertion requires proof.

Finally, I would like to mention two extensions, in addition to those discussed in Section 6. First, in the present paper, I focused on the case of public learning. An important extension would be to formally work out the case in which the consumption experience provides some private information. Second, even with public histories, it would be interesting to consider correlated learning (either in the prior or through the noise). This complicates the efficiency analysis, since the planner’s problem becomes non-separable. Both extensions are left for future research.
Appendix A. Proofs.

Proof of Proposition 1

It is enough to prove the result for Ann. Define an allocation strategy for Ann as a stochastic process taking values on [0,1] which is progressively measurable w.r.t. the filtration generated by \( \{ \theta_t^A \} \). For any allocation strategy \( \lambda \), define

\[
M(\lambda, \theta^A) := \mathbb{E} \left\{ \int_0^\infty e^{-\lambda t} (\lambda_t \mu^* + (1 - \lambda_t)\theta^A_t) \, dt \left| \theta_0^A = \theta^A \right. \right\}. \tag{87}
\]

where \( \{ \theta_t^A \} \) is understood to be the controlled process starting at \( \theta_0^A = \theta^A \) and satisfying the stochastic differential equation

\[
d\theta_t^A = \sqrt{(1 - \lambda_t)^2} \nu(\theta_t^A) d\tilde{Z}_t^A. \tag{88}
\]

This SDE has a unique strong solution for every allocation strategy. It follows from the definitions that \( S^A(\theta^A) = \sup_\lambda M(\lambda, \theta^A) \). Now let \( \theta^* \) be as in the statement and define the solution candidate \( J \in C^1([0,1], \mathbb{R}) \cap C^2([0, \theta^*) \cup (\theta^*, 1], \mathbb{R}) \) by setting

\[
J(\theta^A) := \begin{cases} 
\frac{\mu^*}{r} & \theta \leq \theta^* \\
\theta^A - \left( \frac{\theta^*}{r} - \frac{\mu^*}{r} \right) \left( H(\theta) \right) & \theta > \theta^*. 
\end{cases} \tag{89}
\]

First note that, for every \( \theta^A \in (\theta^*, 1] \), we have

\[
rJ(\theta^A) = \theta^A + \frac{1}{2} \nu(\theta^A)J''(\theta^A) \geq \mu^*. \tag{90}
\]

Moreover, for all \( \theta^A \in [0, \theta^*) \), we have

\[
rJ(\theta^A) = \mu^* \geq \theta^A = \theta^A + \frac{1}{2} \nu(\theta^A)J''(\theta^A). \tag{91}
\]
I will adapt a standard verification argument to prove that \( J(\theta^A) = S^A(\theta^A) \) by showing \( J(\theta^A) \leq S^A(\theta^A) \) and \( J(\theta^A) \geq S^A(\theta^A) \). Although these ideas are well known (see, for instance, Brekke and Oksendal (1991) or Strulovici and Szydlowski (2012)), I include the argument for completeness and because the general results I know assume that the variance of the state process is uniformly bounded away from zero, an assumption which is violated in my model without invalidating the argument.

To show \( J(\theta^A) \leq S^A(\theta^A) \), note that the process stops whenever \( \{\theta_t^A\} \) hits \([0, \theta^*] \). It is then natural to define the stopping time \( \tau^* := \inf\{t > 0 | \theta_t^A \leq \theta^*\} \). Since \( J \) is \( C^2 \) on \((\theta^*, 1]\), we can use Ito’s formula for any fixed \( T > 0 \) to get

\[
\begin{align*}
\mathbb{E}^{-r(T\Lambda^*)}J(\theta_{T\Lambda^*}^A) \\
= J(\theta_0^A) + \int_0^{T\Lambda^*} \mathbb{E}^{-rt} \left( \frac{1}{2} \nu(\theta_t^A)J''(\theta_t^A) - rJ(\theta_t^A) \right) dt \\
+ \int_0^{T\Lambda^*} \mathbb{E}^{-rt} \sqrt{\nu(\theta_t^A)J'(\theta_t^A)} dZ_t^A + \mathbb{E}^{-r(T\Lambda^*)} \left( \frac{\mu^*}{r} \right).
\end{align*}
\]

Using equation (90) which is valid for all \( t < \tau^* \), we get

\[
\begin{align*}
\mathbb{E}^{-r(T\Lambda^*)}J(\theta_{T\Lambda^*}^A) \\
= J(\theta_0^A) - \int_0^{T\Lambda^*} \mathbb{E}^{-rt} \theta_t^A dt + \int_0^{T\Lambda^*} \mathbb{E}^{-rt} \sqrt{\nu(\theta_t^A)J'(\theta_t^A)} dZ_t^A \\
+ \mathbb{E}^{-r(T\Lambda^*)} \left( \frac{\mu^*}{r} \right).
\end{align*}
\]

Rearranging terms and taking expectations conditional on \( \theta_0^A = \theta^A \), we have

\[
J(\theta^A) = \mathbb{E} \left\{ \int_0^{T\Lambda^*} \mathbb{E}^{-rt} \theta_t^A dt + \mathbb{E}^{-r(T\Lambda^*)} \left( \frac{\mu^*}{r} \right) - \mathbb{E}^{-r(T\Lambda^*)}J(\theta_{T\Lambda^*}^A) \bigg| \theta_0^A = \theta^A \right\}
\]

where we used

\[
\mathbb{E} \left\{ \int_0^{T\Lambda^*} \mathbb{E}^{-rt} \sqrt{\nu(\theta_t^A)J'(\theta_t^A)} dZ_t^A \bigg| \theta_0^A = \theta^A \right\} = 0.
\]
Taking limits as $T \to \infty$, we obtain

$$J(\theta^A) = \mathbb{E}\left\{ \int_0^{\tau^*} e^{-rt} \theta^A dt + e^{-r\tau^*} \left( \frac{\mu^*}{r} \right) - e^{-r\tau^*} J(\theta^A) \bigg| \theta_t^A = \theta^A \right\}. \quad (96)$$

Now, it suffices to define

$$\lambda_t^* := \begin{cases} 1 & t \leq \tau^* \\ 0 & t > \tau^* \end{cases} \quad (97)$$

to get

$$J(\theta^A) = M(\lambda^*, \theta^A) \leq S^A(\theta^A). \quad (98)$$

I will now show that $J(\theta^A) \geq S^A(\theta^A)$. Properties (90) and (91) and $J \in C^1([0,1], \mathbb{R})$ imply that, for every $\epsilon > 0$, it is possible approximate the candidate $J$ with a function $\tilde{f}_\epsilon \in C^2([0,1], \mathbb{R})$ satisfying the following conditions:

\begin{enumerate}
  \item[a)] $\sup\{|\tilde{f}_\epsilon(\theta^A) - J(\theta^A)| \big| \theta^A \in [0,1]\} < \epsilon$
  \item[b)] $\forall \theta^A \in [0, \theta^*]: \mu^* \leq r\tilde{f}_\epsilon(\theta^A) + \epsilon$
  \item[c)] $\forall \theta^A \in [\theta^*, 1]: \theta^A + \frac{1}{2}v(\theta^A)\tilde{f}_\epsilon''(\theta^A) \leq r\tilde{f}_\epsilon(\theta^A) + \epsilon$.
\end{enumerate}

It follows that, for every $\gamma \in [0,1]$,

$$\gamma \mu^* + (1 - \gamma) \left( \theta^A + \frac{1}{2}v(\theta^A)\tilde{f}_\epsilon''(\theta^A) \right) \leq r\tilde{f}_\epsilon(\theta^A) + \epsilon. \quad (100)$$

Now pick an arbitrary allocation strategy $\lambda$. Applying Ito’s formula to the $C^2$ function $e^{-rT}\tilde{f}_\epsilon(\theta_T^A)$, we get

$$e^{-rT}\tilde{f}_\epsilon(\theta_T^A) = \tilde{f}_\epsilon(\theta_0^A) + \int_0^T e^{-rt} \left( \frac{1}{2} (1 - \lambda_t)v(\theta_t^A)\tilde{f}_\epsilon''(\theta_t^A) - r\tilde{f}_\epsilon(\theta_t^A) \right) dt$$

$$+ \int_0^T e^{-rt} \sqrt{(1 - \lambda_t)v(\theta_t^A)}\tilde{f}_\epsilon(\theta_t^A) dt. \quad (101)$$

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Using (100), we get

\[ e^{-rT} \tilde{J}_\epsilon(\theta_t^A) \leq \tilde{J}_\epsilon(\theta_0^A) + \int_0^T e^{-rt}(\epsilon - \lambda_t \mu^* - (1 - \lambda_t)\theta_t^A) dt + \int_0^T e^{-rt} \sqrt{(1 - \lambda_t)v(\theta_t^A)\tilde{J}_\epsilon(\theta_t^A)} dt. \]  
(102)

Integrating, rearranging terms and taking expectations conditional on \( \theta_0^A = \theta^A \):

\[ \tilde{J}_\epsilon(\theta^A) + \left( \frac{1 - e^{-rT}}{r} \right) \epsilon \geq \mathbb{E}\left\{ e^{-rT} \tilde{J}_\epsilon(\theta_t^A) + \int_0^T e^{-rt}(\lambda_t \mu^* + (1 - \lambda_t)\theta_t^A) dt \ \big| \theta_0^A = \theta^A \right\}. \]  
(103)

Taking limits as \( T \to \infty \), we get

\[ \frac{\epsilon}{r} \geq \mathbb{E}\left\{ \int_0^\infty e^{-rt}(\lambda_t \mu^* + (1 - \lambda_t)\theta_t^A) dt \ \big| \theta_0^A = \theta^A \right\}. \]  
(104)

It follows that, for every \( \epsilon > 0 \), we have

\[ \tilde{J}_\epsilon(\theta^A) + \frac{\epsilon}{r} \geq M(\lambda, \theta^A). \]  
(105)

Taking limits as \( \epsilon \to 0 \), we get

\[ J(\theta^A) \geq M(\lambda, \theta^A). \]  
(106)

Since \( \lambda \) was arbitrarily chosen,

\[ J(\theta^A) \geq \sup_\lambda M(\lambda, \theta^A) = S^A(\theta^A). \]  
(107)

Having shown that \( J(\theta^A) = S^A(\theta^A) \), the proof is complete.
Proof of Proposition 2

Consider Ann first (the analysis for Bob is symmetric). Given the equilibrium pricing strategy \( \hat{p} \), Ann is indifferent between her stage actions whenever she sees a price \( p^A = \mu^* - \theta^A \). Therefore, buying from the seller is optimal whenever \( p^A \leq \mu^* - \theta^A \). Along the equilibrium path, she will buy from the seller for every \( \theta^A \leq \theta^* \) (i.e. when the seller sets \( p^A = \mu^* - \theta^A \)) and will not buy for \( \theta^A > \theta^* \) (i.e. when the price set by the seller is \( p^A = \mu^* - \theta^* > \mu^* - \theta^A \)). This means that the equilibrium implements the efficient allocation.

Note that if, at \( t = 0 \) in state \( \theta_0^A \), Ann observed \( p_0^A > \mu^* - \theta_0^A \) and expected the inequality \( p_t^A > \mu^* - \theta_t^A \) to hold for a non-negligible (possibly random) period of time \([0, \tau]\), she would get a strictly higher profit by experimenting with the new product. For instantaneous deviations, Ann would be indifferent in terms of total utility, a feature typical of continuous time models.

Now consider the optimality of the seller’s pricing strategy. It is clear that in no equilibrium Ann can get less than \( \theta^A/r \) (what she would expect to get by unconditional continuation). Since that is exactly what she is getting when sold, the pricing strategy for \( \theta \leq \theta^* \) must be optimal. On the other hand, selling for \( \theta > \theta^* \) is not optimal since if it was, it would also be optimal for the planner to have Ann consuming the seller’s product. In other words, the solution to the planner’s problem shows that the seller’s expected discounted value of waiting for Ann to reach \( \theta^* \) and start buying at price \( p_t^A = \hat{p}^A(\theta^*) = \mu^* - \theta^* \) exceeds the value of attempting to attract Ann at her current optimistic state.\[\blacksquare\]
Proof of Proposition 3

Given that consumers are informationally small, the cutoff price $\tilde{p}^i = \mu^* - \theta^i$ is obviously optimal.

To verify optimality for the seller, we note that the marginal buyer is always indifferent. Hence, the question reduces to whether the marginal buyer is optimally chosen across the state space. In the region, $R_2 \cup R'_2 \cup R_3 \cup R'_3$ with $\min\{\theta^A, \theta^B\} \leq \theta^c$, the seller is always choosing the switching cutoffs optimally by solving her optimal stopping problem. Moreover, along the diagonal, there is no need for price discrimination, so it is optimal to target both consumers, as long as it is optimal to target any. Note that, in fact, if $\theta^A \geq \theta^B \in [\theta^c, \theta^*]$ or $\theta^B \geq \theta^A \in [\theta^c, \theta^*]$, both Ann and Bob are too optimistic about the new product and the only situation in which the seller can profitably target both simultaneously is when beliefs are symmetric (otherwise it is better to target only the less optimistic of the two). Moreover, given the expectations of consumers (encoded in their equilibrium value functions), increasing the price cannot increase the seller’s profits because it will always ensure the loss of the marginal consumer she is optimally choosing to serve. Reducing the price can do the seller no good either (even if it attracts a consumer that was experimenting, this cannot be optimal under optimally chosen cutoffs).

Finally, it is intuitive that selling the product in $R_4$ cannot be profit maximizing since it requires the seller to reduce its prices while simultaneously decreasing total social surplus. To see this more formally, suppose that $\theta^B \geq \theta^A$. Since Bob is going to experiment anyway, the seller’s incentives are not changed compared to what happens when $\theta^B = 1$. Hence, the solution to her optimal stopping problem does not change and is given by $\tilde{\theta}(\theta^B) = \theta^*$ for all $\theta^B \in (\theta^*, 1]$. A symmetric argument covers the case $\theta^B \leq \theta^A$ and hence every $\theta \in R_4$.

Since the pricing strategy is optimal in every region, we conclude that $(\hat{p}, \tilde{p})$ is a MPE as claimed $\blacksquare$
Proof of Proposition 4

Suppose that the condition is violated. Note that if $\theta_0 \in R_1 \cup R_2 \cup R'_2$, the equilibrium strategies induce beliefs to remain at the initial state forever. Moreover, if $\theta_0 \in \overline{R_4}$, beliefs cannot exit $\overline{R_4}$ since, at the boundary, the consumer with lower $\theta^i$ never experiments. Finally, when $\theta^A = 1$ or $\theta^B = 1$, we are essentially in the price discrimination case. Since the equilibrium prescribes efficient product choices in all these regions, one direction is proved.

For the other direction, suppose that the condition is satisfied. Note that, if $\theta_0 \in R_3$ and $\theta^A_0 < 1$, there is positive probability of $\theta_t$ crossing through the state $(\theta^*, \theta^B_0)$. Similarly, if $\theta_0 \in R_3'$ and $\theta^B_0 < 1$, there is positive probability of crossing through $(\theta^A_0, \theta^*)$. Hence, either the equilibrium already prescribes an inefficient action on $\theta_0$ or there is positive probability of $\theta_t$ entering a region in which an inefficient action is prescribed.
Proof of Proposition 5

Let \( \theta_0 \) be any initial state. Consider the set of rest points for the dynamics of the belief process \( \{ (\theta^A_t, \theta^B_t) \} \):

\[
\Lambda := \{(\theta^A, \theta^B) \in [0,1]^2 | 1\{\hat{p}(\theta) \leq \bar{p}^A(\theta)\}v(\theta^A) + 1\{\hat{p}(\theta) \leq \bar{p}^B(\theta)\}v(\theta^B) = 0\} \\
= R_0 \cup R_1 \cup R'_1 \cup R_3 \cup \{(0,1), (1,0), (\theta^c, 1), (1, \theta^c), (1,1)\}. \tag{108}
\]

Note that \( \Lambda \) is closed in \([0,1]^2\). Hence, the hitting time \( \tau := \inf\{t > 0 | \theta_t \in \Lambda\} \) is a stopping time w.r.t. \( \{\mathcal{F}_t\} \). Since \( \{\theta_t\} \) is a \( \{\mathcal{F}_t\} \) martingale, the stopped process \( \{\theta_{t\wedge \tau}\} \) is also a \( \{\mathcal{F}_t\} \) martingale (see Theorem 3.22 in [4]). Therefore, \( \{\theta_{t\wedge \tau}\} \) converges almost surely to some random variable \( \bar{\theta}_\infty \) as \( t \to +\infty \) by the martingale convergence theorem (see Theorem 3.15 in [4]). Clearly, \( \bar{\theta}_\infty \in \Lambda \) almost surely.

Note that the equilibrium prescribes an efficient outcome for every point in \( \Lambda \). Hence, it only remains to show that \( \Lambda \) is reached in finite time almost surely whenever the initial state lies in the inefficient region \( \text{int } R_3 \cup \text{int } R'_3 \). By symmetry, it suffices to consider the case \( \theta_0 \in \text{int } R_3 \) (that is \( \theta^A_0 \geq \bar{\theta}(\theta^B_0) \) and \( \theta^B_0 < \theta^* \)). Note that \( \theta^B_t = \theta^B_0 \) for all \( t \), since \( \hat{q}^B(\theta_t) = 1 \) whenever \( \theta_t \in R_2 \cup R_1 \). If \( \mu^A = 0 \), then \( \theta^A_t \) will have negative drift bounded away from zero in \( R_2 \). As a consequence, the probability that \( \theta^A_t = \bar{\theta}(\theta^B_0) \) in finite time is 1. If, on the contrary, \( \mu^A = 1 \), then the drift will be positive and, with probability 1, either \( \theta^A_t = \bar{\theta}(\theta^B_0) \) in finite time or \( \theta^A_t \to 1 \). Moreover, since \( \theta^* < 1 \), \( \theta^A_t \to 1 \) implies that \( \theta^A_t \in [\theta^*, 1] \) for all sufficiently large \( t \).

Since \( \mu^A \in \{0,1\} \) with probability 1 and the claim is true conditional on \( \mu^A = 0 \) and \( \mu^A = 1 \), the theorem is proved \( \blacksquare \)
Proof of Proposition 6

The Lebesgue measure of the inefficient area is given by

\[
L(\mu^*, r, \sigma) := 2 \int_0^{\theta^c} \left( \theta^* - \bar{\theta}(\theta) \right) d\theta + (\theta^* - \theta^c)^2
\]

\[
= 2\theta^*\theta^c - 2 \int_0^{\theta^c} \bar{\theta}(\theta) d\theta + (\theta^* - \theta^c)^2.
\]  

(109)

Note that, defining \(a := \alpha \mu^*, \ b := \alpha\) and \(c := \mu^* - 2\beta\)

\[
\bar{\theta}(\theta) = \frac{\alpha(\mu^* + \theta)}{\mu^* + \theta - 2\beta} \equiv \frac{a + b\theta}{c + \theta}
\]  

(110)

and

\[
\int_0^{\theta^c} \left( \frac{a + b\theta}{c + \theta} \right) d\theta = (bc - a)(\ln c - \ln(c + \theta^c)) + b\theta^c
\]

\[
= 2\alpha\beta \ln \left(1 + \frac{\theta^c}{\mu^* - 2\beta}\right) + a\theta^c.
\]  

(111)

Hence,

\[
L(\mu^*, r, \sigma) = (\theta^*)^2 + (\theta^c)^2 - 4\alpha\beta \ln \left(1 + \frac{\theta^c}{\mu^* - 2\beta}\right) - 2a\theta^c.
\]  

(112)

Note that \(\lim_{\sigma \to \infty} \theta^* = \lim_{\sigma \to \infty} \theta^c = \mu^*\) and

\[
\lim_{\sigma \to \infty} \left(4\alpha\beta \ln \left(1 + \frac{\theta^c}{\mu^* - 2\beta}\right) + 2a\theta^c\right) = \frac{3}{2} (\mu^*)^2.
\]  

(113)

It follows that

\[
\lim_{\sigma \to \infty} L(\mu^*, r, \sigma) = \frac{(\mu^*)^2}{2}.
\]  

(114)

Since \(L(\mu^*, r, \sigma)\) depends on \(r\) and \(\sigma\) only through \(r\sigma^2\), we have

\[
\lim_{\sigma \to \infty} L(\mu^*, r, \sigma) = \lim_{r \to \infty} L(\mu^*, r, \sigma).
\]  

(115)

This completes the proof.$\blacksquare$
References


