Abstract

I study the problem of a durable goods monopolist who lacks commitment power and whose marginal cost of production varies stochastically over time. Time-varying costs modify the results on the Coase conjecture. When the distribution of consumer valuations is discrete, the monopolist is able to exercise market power. Moreover, there is inefficient delay in equilibrium. In contrast, with a continuum of types the monopolist is unable to extract additional rents from consumers with higher valuations, and the market outcome is first best efficient. The model is set up in continuous time and the monopolist’s marginal cost evolves as a diffusion process. Continuous time methods lead to a tractable characterization of the equilibrium.

* I am indebted to Faruk Gul, Wolfgang Pesendorfer and Sylvain Chassang for guidance, encouragement and support. I also thank Dilip Abreu, Jan De Loecker, Myrto Kalouptsidi, Stephen Morris, Satoru Takahashi, Ben Brooks, Edoardo Grillo, Leandro Gorno and seminar participants at Princeton, Duke, WUSTL, Northwestern/MEDS, Boston University, Bocconi, Toulouse School of Economics, Arizona State and Rochester for helpful comments. All remaining errors are my own. Address: Boston University, Department of Economics, 270 Bay State Road, Boston, MA 02215. E-mail: jmortner@gmail.com.
1 Introduction

Consider the problem of a monopolist who produces a durable good and who cannot commit to a path of prices. For settings in which production costs do not change over time, Coase (1972) argued that such a producer would not be able to sell at the static monopoly price. After selling the initial quantity, the monopolist has the temptation to reduce prices to reach consumers with lower valuation. This temptation leads the monopolist to continue cutting prices after each sale. Forward looking consumers expect prices to fall, so they are unwilling to pay a high price. Coase conjectured that these forces would lead the monopolist to post an opening price arbitrarily close to marginal cost. The monopolist would then serve the entire market “in the twinkling of an eye”, and the market outcome would be efficient. The classic papers on durable goods monopoly (i.e., Stokey, 1981, and Gul, Sonnenschein and Wilson, 1986) provide formal proofs of the Coase conjecture: as the period length goes to zero, the monopolist’s opening price converges to the lowest consumer valuation. In the limit, all buyers trade immediately, the market outcome is efficient and the monopolist is unable to extract additional rents from those buyers with higher valuations.

The purpose of this paper is to study the problem of a durable goods monopolist who lacks commitment power and whose cost of production varies stochastically over time. The assumption that costs are subject to stochastic shocks is natural in many markets. Time-varying costs may arise as a result of changes in input prices. For instance, high-tech firms face uncertain and time varying costs, partly because the prices of some of their key inputs tend to fall over time, and partly due to fluctuations in the prices of the raw materials that they use.\(^1\) Changes in exchange rates will also lead to time-varying costs if the monopolist sells an imported good or if she uses imported inputs.

The model is set up in continuous time and the monopolist’s marginal cost evolves as a diffusion process. Continuous time methods are especially suitable to perform the option value calculations that arise with time-varying costs, allowing me to obtain a tractable characterization of the equilibrium. The model delivers simple expressions for the prices at which buyers are willing to trade, allowing for the computation of profit margins as a function of costs and the level of market penetration.

With time-varying costs, serving the entire market immediately is in general not efficient. The reason for this is that time-varying costs introduce an option value of delaying trade. The efficient outcome in this setting is that the monopolist serves consumers with valuation \(v\) the first time costs fall below a threshold \(z_v\). The threshold \(z_v\) is decreasing in the valuation \(v\).

\(^1\)See Conlon (2010) for evidence on the evolution of production costs in the LCD TV industry.
Based on this observation, I propose the following generalization of the Coase conjecture for markets with time-varying costs. Given a distribution of consumer valuations, I define an outcome to be *Coasian* if (i) it is pareto efficient, and (ii) the monopolist is unable to extract additional rents from those consumers with higher valuations. Note that the second condition does not require the monopolist to sell to all consumers at the same time or at the same price; in fact, doing so would violate efficiency.

In this paper, I show that the market outcome fails to be Coasian when costs are time-varying and the distribution of consumer valuations is discrete. With discrete types, the monopolist is able to extract rents from consumers with higher valuations, since she can truthfully commit to delay trade with buyers with lower types until costs are low enough. Moreover, there is inefficient delay in equilibrium. In contrast, the market outcome is Coasian when there is a continuum of types. In this case, the monopolist has an incentive to serve the next buyer arbitrarily soon after her last sale. As a result, the monopolist is no longer able to extract rents from consumers with higher valuations, and the market outcome is efficient. One possible interpretation of these results is that a dynamic monopolist with time-varying costs can obtain more profits in markets in which there is a clear segmentation among buyers. Examples include a monopolist selling in different geographical locations, or in the case of intermediate durable goods, a monopolist selling to firms in different industries.

Coase’s original arguments illustrate how commitment problems may prevent a monopolist producer of a durable good from exercising market power. The results in this paper show that these forces are more general than what Coase described. In particular, these forces do not rely on serving the entire market immediately, nor on serving every buyer at the same price. In markets with time-varying costs, to attain efficiency and zero rent extraction it is enough that the monopolist cannot credibly commit to delay trade from one sale to the next.

To see how time-varying costs modify the results on the Coase conjecture, consider a setting with two types of buyers: high types, with valuation \( v_H \), and low types, with valuation \( v_L < v_H \). After high types buy and leave the market, the monopolist’s problem is to choose when to sell to the remaining low type buyers. When costs do not change over time, it is optimal for the monopolist to sell to low types immediately after selling to high types. This is the force behind the Coase conjecture: high types are not willing to pay a high price, since they expect prices to fall rapidly after they buy. With time-varying costs, the monopolist will only sell to low types when costs fall below a threshold \( z_L \). High types know that it will take a non-negligible amount of time for prices to fall when costs are above \( z_L \), so they are willing to pay a higher price. In a sense, time-varying costs endogenously provide commitment power
to the monopolist.

The equilibrium dynamics with two types of buyers are as follows. If costs are initially above a threshold $\bar{x} > z_L$, the monopolist first sells to all high type buyers, and then sells to all low types when costs fall below $z_L$. If costs are below a threshold $\underline{x} < z_L$, the monopolist sells immediately to high and low types and the market closes. Finally, when costs initially lie between $\underline{x}$ and $\bar{x}$ the monopolist sells to high types gradually over time and market penetration increases continuously. The equilibrium is inefficient when costs lie in this intermediate region, since the first-best outcome in this case is that all high types trade immediately.

I study markets with a continuum of valuations by analyzing a sequence of discrete models that approximates a model with a continuum of types. I show that the equilibrium outcome becomes Coasian as types become a continuum. In the limit, the monopolist serves consumers sequentially as costs decrease, exactly at the time that maximizes total surplus, and she cannot extract rents from consumers with higher valuations. Intuitively, the monopolist loses all commitment power when the gap between valuations becomes vanishingly small, since she now has an incentive to serve the next buyer arbitrarily soon after her last sale. As a result, she is no longer able to extract rents and the equilibrium outcome is efficient.

This paper contributes to the growing literature that uses continuous time methods to analyze strategic interactions.\(^2\) The analysis of games in continuous time presents technical difficulties (i.e., Simon and Stinchcombe, 1989). One of these difficulties is that subgame perfection has less bite in continuous time durable goods monopoly games than in their discrete time counterparts. In discrete time, buyers incur a fixed cost of delay if they choose not to buy at the current price, since they must wait one time period for the seller to post a new price. This cost of delay imposes strong restrictions on the strategies that buyers use in a subgame perfect equilibrium (SPE). In contrast, buyers do not face a fixed cost of delay when the game is in continuous time, since they can always accept a new price within an arbitrarily short period of time. As a result, the continuous time game has equilibria that could never arise in its discrete time counterpart.\(^3\) In this paper, I deal with this by building into the definition of equilibrium the intuitive restriction that buyers accept prices that leave them indifferent between trading at that price or delaying trade until the monopolist’s next sale. These restrictions that I impose would necessarily hold in any SPE of the discrete time

\(^2\)For instance, continuous time methods have been used to study the provision of incentives in dynamic settings (Sannikov, 2007, 2008), political campaigns (Gul and Pesendorfer, 2012) and dynamic markets for lemons (Daley and Green, 2012).

\(^3\)This is closely related to the multiplicity of equilibrium outcomes that arises in continuous time bilateral bargaining games in which there are no restrictions on the timing of offers and counteroffers. See, for instance, Bergin and MacLeod (1993) and the discussion in Perry and Reny (1993, pp. 66-67).
1.1 Related literature

The literature on durable goods monopoly has identified different ways in which a dynamic monopolist can exercise market power. For instance, a durable goods monopolist can ameliorate her lack of commitment power by renting her good rather than selling it (Bulow, 1982), or by introducing best-price provisions (Butz, 1990). The Coase conjecture also fails when the monopolist faces capacity constraints (Kahn, 1986), and McAfee and Wiseman, 2008), or when consumers use non-stationary strategies (Ausubel and Deneckere, 1989, and Sobel, 1991). The current paper studies the problem of a durable goods monopolist whose production costs change over time and identifies a new setting in which such a seller can exercise market power: time-varying costs provide commitment power to the monopolist when types are discrete, allowing her to extract rents from buyers with higher valuations.4

This paper also shares some features with models of bargaining with one-sided incomplete information (i.e., Fudenberg, Levine and Tirole, 1985). Deneckere and Liang (2006) study a bargaining game in which the valuation of the buyer is correlated with the cost of the seller (see also Evans, 1989, and Vincent, 1989). They show that there are recurring bursts of trade in equilibrium, with short periods of high probability of agreement followed by long periods of delay. In the current paper, there are also recurring bursts of trade when types are discrete. For instance, with two types of buyers the monopolist first sells to all high types when costs are initially large, and then sells to low types when costs fall below $z_L$.

Fuchs and Skrzypacz (2010) study a one-sided incomplete information bargaining game in which a new trader may arrive according to a Poisson process. The payoff that the seller and the buyer get upon an arrival depends on the buyer’s valuation for the seller’s good; for instance, upon arrival the seller may run a second price auction between the original buyer and the new trader. Fuchs and Skrzypacz (2010) show that the seller is unable to extract rents in this setting: her inability to commit to a path of offers drives her profits down to her outside option of waiting until the arrival of a new buyer. Moreover, the possibility of arrivals leads to inefficient delays, with the seller slowly screening out high type buyers. In the current paper, the monopolist is also unable to extract rents when there is a continuum of types. However, the equilibrium outcome is efficient in this setting, with the seller serving

4Other papers study dynamic monopoly models in non-stationary environments. Stokey (1979) solves the full commitment pricing path of a durable good monopolist when costs evolve deterministically over time. Board (2008) characterizes the full commitment strategy of a durable good monopolist when incoming demand varies over time. Biehl (2001) studies a setting in which the buyers’ valuations are subject to shocks.
the different buyers at the point in time that maximizes total surplus.

The possibility of arrivals in Fuchs and Skrzypacz (2010) introduces interdependencies in the net valuations of the buyer and the seller, making their setting similar to the one in Deneckere and Liang (2006). In a different paper, Fuchs and Skrzypacz (2012) consider the model in Deneckere and Liang (2006) with a continuum of types. They show that the equilibrium of this model converges to the outcome in Fuchs and Skrzypacz (2010) as the gap between the seller’s lowest cost and the buyer’s lowest valuation converges to zero. The seller is therefore unable to extract rents from the buyer in this gapless limit. In the current paper, the monopolist loses the ability to extract rents as valuations become a continuum. The difference, however, is that this result holds for any model with a continuum of types, regardless of the lowest consumer valuation (i.e., regardless of the size of the gap).

2 Model

A monopolist faces a unit measure of non-atomic consumers indexed by $i \in [0, 1]$. Consumers are in the market to buy one unit of the monopolist’s good. Time is continuous and consumers can make their purchase at any time $t \in [0, \infty)$. The valuations of the consumers are defined by a non-increasing and left-continuous function $f : [0, 1] \to [\underline{v}, \overline{v}]$ with $\overline{v} > \underline{v} > 0$; consumer $i$’s valuation for the good is $f(i)$. Therefore, consumers with a lower index have a weakly higher valuation. Consumers and the monopolist are risk-neutral expected utility maximizers and discount future payoffs at rate $r > 0$. I assume that $f$ is a step function taking $n$ values $v_1, \ldots, v_n$, with $v_1 < \ldots < v_n$. For $k = 1, \ldots, n$, let $\alpha_k = \max\{i \in [0, 1] : f(i) = v_k\}$; i.e., $\alpha_k$ is the highest indexed consumer with valuation $v_k$. Section 6 considers the case in which $f$ approximates a continuous function.

Let $B = \{B_t, \mathcal{F}_t : 0 \leq t < \infty\}$ be a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, where $\{\mathcal{F}_t : 0 \leq t < \infty\}$ is the completion of the filtration generated by the Brownian motion. The Brownian motion drives the monopolist’s marginal cost $x_t$ as

$$dx_t = \mu x_t dt + \sigma x_t dB_t,$$

with $x_0 = x > 0$, $\sigma > 0$ and $|\mu| < r$. At time $t$ the monopolist can produce any desired quantity at marginal cost $x_t$. The assumption that $\mu < r$ guarantees that the monopolist will always produce on demand: under this condition it is never optimal for the monopolist to produce when costs are low to sell in the future when costs are high. On the other hand,

\footnote{The assumption that $x_t$ evolves as (1) guarantees that $x_t > 0$ for all $t \geq 0$.}
the assumption that $\mu > -r$ guarantees that the monopolist will serve consumers in finite time. The constants $\mu$ and $\sigma$ measure the expected rate of change of $x_t$ and the volatility of $x_t$, respectively. The process $x_t$ is publicly observable and its underlying structure is common knowledge: monopolist and consumers commonly know that $x_t$ evolves as (1).

A (stationary) strategy for consumer $i \in [0,1]$ is a function $P: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that describes the price that $i$ is willing to pay for the good given any level of costs. Suppose consumer $i$ is still in the market at time $t$. Then, under strategy $P(\cdot)$ consumer $i$ purchases the good at time $t$ if and only if the price that the monopolist charges is weakly lower than $P(x_t)$.

Let $P = P(x,i)$ be a strategy profile for the consumers, with $P(\cdot,i)$ denoting the strategy of $i \in [0,1]$. In any equilibrium, the strategy profile $P(x,i)$ must satisfy the skimming property: for all $i < j$, $P(x,i) \geq P(x,j)$ for all $x$. That is, buyers with higher valuations are willing to pay higher prices. The reason for this is that it is more costly for buyers with higher valuation to delay their purchase: if a buyer with valuation $v$ finds it weakly optimal to buy at some time $t$ given a future path of prices, then buyers with valuation $v' > v$ find it strictly optimal to buy at $t$. For technical reasons, I restrict attention to strategy profiles such that $P(x,i)$ is left-continuous in $i$ and continuous in $x$. This restriction guarantees that payoffs are well defined.

The skimming property implies that at any time $t$ there exists a cutoff $q_t \in [0,1]$ such that consumers $i \leq q_t$ have already left the market, while consumers $i > q_t$ are still in the market. The cutoff $q_t$ describes the level of market penetration at time $t$. At each time $t$, the level of market penetration and the monopolist’s marginal cost describe the payoff relevant state of the game. Clearly, the initial level of market penetration $q_0$ is equal to 0.

Given a strategy profile $P$, the monopolist chooses a path of prices to maximize her profits. Since $P$ satisfies the skimming property, by setting a price $p$ the monopolist effectively chooses the level of market penetration: if the monopolist sets price $p$ at time $t$, there will be a $q \in [0,1]$ such that $P(x_t,i) \geq p$ if and only if $i \leq q$. Moreover, the monopolist will find it optimal to charge a price $P(x_t,q)$ if consumer $q$ is the marginal buyer at time $t$. Thus, I can alternatively specify the monopolist’s problem as choosing a non-decreasing process $\{q_t\}$ with $q_0 = 0$ and $q_t \leq 1$ for all $t$, describing the level of market penetration at any time $t$. With this specification, under strategy $\{q_t\}$ the seller charges $P(x_t,q_t)$ at every time $t$, and at this price all buyers $i \leq q_t$ who are still in the market buy.

**Remark 1** Throughout the paper, I maintain the standard assumption that the monopolist cannot ration consumers: at every point in time the monopolist sells to all buyers who are willing to purchase at the current price. This assumption implies that the process $\{q_t\}$ must
satisfy the following condition. Suppose that $P$ is such that $P(x, i) = p$ for all $i \in [l, h] \subseteq [0, 1]$ and that $\{q_t\}$ is such that the monopolist makes sales at time $t$ whenever $q_t \in [l, h]$ and $x_t = x$. Note that in order to sell at time $t$, the monopolist has to set a price of at most $P(x, q_t) = p$; and at this price all consumers $i \in [q_t, h]$ will buy the good. Thus, in this case the process $\{q_t\}$ must be such that $dq_t \geq h - q_t$; i.e., $\{q_t\}$ must jump at time $t$. Formally, for every $t$ such that $P(x_t, \cdot)$ is constant in an interval $[q_t, h]$ with $h > q_t$, if $dq_t > 0$ then it must be that $dq_t \geq h - q_t$.

**Monopolist’s problem:** Given a strategy profile $P$ of the consumers, a strategy for the seller is an $\mathcal{F}_t$-progressively measurable process $\{q_t\}$ satisfying the conditions in Remark 1 such that $q_0 = 0$, $q_t$ is non-decreasing with $q_t \leq 1$ for all $t$, and $\{q_t\}$ is right-continuous with left-hand limits. Let $A^P$ denote the set of all such processes. Given a strategy profile $P$ of the consumers and a strategy $\{q_t\} \in A^P$, the monopolist’s profits are

$$
\Pi = E \left[ \int_0^\infty e^{-rt} (P(x_t, q_t) - x_t) dq_t \right].
$$

(2)

Let $\Pi(x, q)$ denote the monopolist’s future discounted profits conditional on the current state being $(x, q)$, and let $A^P_{q,t}$ denote the set of processes in $A^P$ such that $q_t = q$. Then, the monopolist’s payoffs conditional on state at time $t^-$ being $(x, q)$ are

$$
\Pi(x, q) = \sup_{\{q_t\} \in A^P_{q,t}} E \left[ \int_t^\infty e^{-r(s-t)} (P(x_s, q_s) - x_s) dq_s \right] _{\mathcal{F}_t}.
$$

(3)

Condition (3) is the requirement that the monopolist’s strategy $\{q_t\}$ is subgame perfect (i.e., time-consistent), since the strategy $\{q_t\}$ must be optimal at every point in time.

**Consumer’s problem:** Given a strategy $\{q_t\}$ of the monopolist and a strategy profile $P$ of the consumers, the path of prices is $\{P(x_t, q_t)\}$. The strategy $P(x, i)$ of each consumer $i$ must be optimal given the path of prices $\{P(x_t, q_t)\}$: the payoff that consumer $i$ gets from buying at the time strategy $P(x, i)$ tells her to buy must be weakly larger than what she would get from purchasing at any other point in time. Formally, for any time $t$ before consumer $i$ buys, it must be that $f(i) - p \leq \sup_{\tau} E_t[e^{-r(t-\tau)}(f(i) - P(x_\tau, q_\tau))]$ for any $p > P(x, i)$, and

---

6. These continuity requirements on $\{q_t\}$ together with the continuity requirements on $P(x, i)$ guarantee that the integrals in (2) and (3) are well-defined. Indeed, when the filtration is the one generated by the Brownian motion, any right-continuous or left-continuous process is predictable, and hence integrable with respect to a semi-martingale satisfying the properties of $q_t$ (see, for instance, Klebaner, 2005, Chapter 8).

7. Note that the profits $\Pi(x, q)$ are conditional on the state at time $t^-$. The reason for this is to preserve the right-continuity of $\{q_t\}$, since this process may jump at time $t$. 

7
\( f (i) - p \geq \sup_{\tau} E_t [e^{-r(\tau-t)} (f (i) - P (x_\tau, q_\tau))] \) for any \( p \leq P (x, i) \).

I impose two additional conditions on the strategies of the consumers. First,

\[ \forall i \text{ such that } f (i) = v_1, P (x; i) = v_1 \text{ for all } x. \]  \hspace{1cm} (4)

In words, all consumers with the lowest valuation are willing to pay a price equal to their valuation. The second condition I impose is as follows. Fix a strategy profile \((P, \{q_t\})\). Recall that \( \alpha_k \) is the highest indexed consumer with valuation \( v_k \). For \( k = 1, \ldots, n \), let \( \tau (k) \) denote the (possibly random) time at which the monopolist starts selling to consumers with valuation \( v_k \), i.e., \( \tau_k = \inf \{t : q_t > \alpha_{k-1}\} \). Then, for \( k = 2, \ldots, n \),

\[ v_k - P (x, \alpha_k) = E \left[ e^{-r(\tau_{k-1}-t)} (v_k - P (x_{\tau_{k-1}}, q_{\tau_{k-1}})) \right| \mathcal{F}_t], \]  \hspace{1cm} (5)

whenever the state at \( t \) is \((x, \alpha_k)\). Equation (5) is an incentive compatibility condition stating that the price consumer \( \alpha_k \) is willing to pay must leave her indifferent between buying at that price or waiting and buying at the price at which the monopolist starts selling to consumers with valuation \( v_{k-1} \). The paragraphs below provide a justification for these two conditions.

**Definition 1** A strategy profile \((P, \{q_t\})\) is an equilibrium if:

(i) \( \{q_t\} \) is optimal for all states \((x, q)\) given \( P \),

(ii) For each \( i \), \( P (x, i) \) is optimal given \( \{q_t\} \) and \( P \), and

(iii) \( P \) satisfies conditions (4) and (5), given \( \{q_t\} \).

Conditions (i) and (ii) above require that the strategies of the seller and the buyers be optimal. Condition (iii), on the other hand, imposes additional restrictions on the strategies of the buyers. The motivation behind these additional restrictions is as follows. In durable goods markets, buyers can delay trade and make their purchase at a future time. This ability to delay trade gives buyers an outside option, since a buyer who rejects the current price always has the opportunity to trade in the future, possibly at a lower price. Essentially, condition (iii) restricts the surplus that the buyers extract to be exactly equal to the value of this endogenous outside option to delay trade. For instance, equation (5) restricts the highest indexed buyer with valuation \( v_k \) to accept prices that leave her indifferent between trading at that price or delaying trade until the monopolist’s next sale. On the other hand, the value of delaying trade for buyers with valuation \( v_1 \) is zero, since there will be no further
sales after this group of consumers buys. Therefore, buyers with valuation \( v_1 \) should accept prices that leave them with zero surplus, which is exactly what equation (4) implies.\(^8\) These restrictions are not redundant, as there are strategy profiles that satisfy conditions (i) and (ii) in definition 1, but that violate condition (iii).\(^9\)

### 3 First-best outcome

This section computes the first-best outcome. Recall that the function \( f \) describing the consumer valuations is a step function taking values \( v_1 < \ldots < v_n \). To compute the efficient outcome, consider first the problem of choosing the surplus maximizing time at which to serve a homogeneous group of buyers with valuation \( v_k \). This problem is given by

\[
V_k(x) = \sup_{\tau \in T} E\left[ e^{-r\tau} (v_k - x_\tau) \mid x_0 = x \right],
\]

where \( T \) is the set of stopping times. Let \( \lambda \) be the negative root of \( \frac{1}{2} \sigma^2 \lambda (\lambda - 1) + \mu \lambda = r \), and for \( k = 1, \ldots, n \) let \( z_k = \frac{-\lambda}{1-\lambda} v_k \).

**Lemma 1** The stopping time \( \tau_k = \inf \{ t : x_t \leq z_k \} \) solves (6). Moreover,

\[
V_k(x) = \begin{cases} 
(v_k - z_k) \left( \frac{x}{z_k} \right)^\lambda & x > z_k, \\
v_k - x & x \leq z_k.
\end{cases}
\]

**Proof.** See Appendix A.1. \( \blacksquare \)

Lemma 1 captures the option value that arises when costs vary stochastically over time. The total surplus from serving a group of buyers with valuation \( v_k \) is maximized by waiting until costs fall below the threshold \( z_k \). The threshold \( z_k \) is increasing in \( \mu \) and decreasing in \( \sigma \), so it is optimal to wait longer when costs fall faster or when they are more volatile. By Lemma 1, the first-best outcome is that the monopolist serves consumers with valuation \( v_k \) at time \( \tau_k \). For instance, if \( x_0 > z_n \), under the first-best outcome the monopolist serves

---

\(^8\)One can easily show that the restrictions that condition (iii) imposes on the buyers’ strategies would hold in any SPE of a discrete time version of this paper’s model. The reason for this is that, in discrete time, buyers incur a fixed cost of delay if they choose not to buy at the current price, since they must wait one time period for the monopolist to post a new price. This fixed cost of delay imposes strong restrictions on the strategies that buyers use in a SPE; among them, the restrictions in condition (iii).

\(^9\)For instance, the strategy profile under which the monopolist always sells at a price equal to marginal cost, and in which each consumer buys the monopolist’s good at the optimal time (given that the monopolist always sells at price \( x_t \)) satisfies conditions (i) and (ii) in definition 1, but doesn’t satisfy condition (iii).
consumers with valuation \( v_n \) the first time \( x_t = z_n \). After that, the monopolists serves to consumers with valuation \( v_{n-1} \) the first time \( x_t = z_{n-1} \), and so on.

4 Markets with two types of consumers

In this section, I study markets with two types of buyers. Section 4.1 derives the equilibrium for such markets, and Section 4.2 presents the more salient features of the equilibrium.

4.1 Characterization of the equilibrium

Suppose there are two types of buyers in the market: high types with valuation \( v_2 \), and low types with valuation \( v_1 \). Let \( \alpha \in (0,1) \) be the fraction of high type buyers, so that \( f(i) = v_2 \) for all \( i \in [0,\alpha] \) and \( f(i) = v_1 \) for all \( i \in (\alpha,1] \).

By equation (4), consumers with valuation \( v_1 \) will only buy when the price equals \( v_1 \). That is, \( \forall i \in (\alpha,1], P(x,i) = v_1 \) for all \( x \). For any \( q \geq \alpha \), let \( \Pi(x,q) \) denote the monopolist’s profits when level of market penetration is \( q \) and costs are \( x \). Note that at such a state only consumers with valuation \( v_1 \) remain in the market. Since all consumers with valuation \( v_1 \) buy at the same time, at any state \( (x,q) \) with \( q \geq \alpha \) the problem of the monopolist is to optimally choose the time at which to sell to all consumers remaining in the market:

\[
\Pi(x,q) = (1-q) \sup_r E[e^{-r_1} (v_1 - x_r)|x_0 = x].
\]

By Lemma 1, the solution to this problem is \( \tau_1 = \inf \{t : x_t \leq z_1 \} \). Thus, \( \Pi(x,q) = (1-q) V_1(x) \) for all \( q \in [\alpha,1] \).

Consider next the case in which the level of market penetration is \( q \in [0,\alpha) \), so there are \( \alpha - q \) high types remaining in the market. To study equilibrium at these states, I proceed in two steps. First, I establish a lower bound on the monopolist’s profits. Then, I show that the monopolist’s equilibrium profits are exactly equal to this lower bound.

Consider the strategy \( P(x,\alpha) \) of consumer \( \alpha \), the highest indexed consumer with valuation \( v_2 \). After consumer \( \alpha \) buys and leaves the market, the monopolist will sell to low type buyers the first time costs fall below the threshold \( z_1 \), and will charge them a price equal to \( v_1 \). Therefore, by equation (5), in any equilibrium \( P(x,\alpha) \) must satisfy

\[
P(x,\alpha) = v_2 - E [e^{-r_1} (v_2 - v_1)|x_0 = x].
\]

That is, when costs are equal to \( x \), consumer \( \alpha \) must be indifferent between buying at price \( P(x,\alpha) \) or waiting until costs fall below \( z_1 \) and obtaining the good at price \( v_1 \). Equation (8) highlights the commitment power that time-varying costs provide to the monopolist. When
\[ P(x, \alpha) \]

\[ x > z_1, \text{ consumer } \alpha \text{ knows that the monopolist won’t lower her price to } v_1 \text{ until costs fall below } z_1, \text{ so she is willing to pay strictly more than } v_1 \text{ (see Figure 1 for a plot of } P(x, \alpha)). \]

**Lemma 2**  \( P(x, \alpha) - x > V_1(x) \text{ for all } x \in (z_1, z_2]. \) Moreover,

\[
P(x, \alpha) = \begin{cases} 
  v_2 - (v_2 - v_1) \left( \frac{x}{z_1} \right)^{\lambda} & x > z_1, \\
  v_1 & x \leq z_1.
\end{cases}
\]  \( (9) \)

**Proof.** See Appendix A.1. \( \blacksquare \)

Since the strategy profile of the buyers satisfies the skimming property, it must be that \( P(x, i) \geq P(x, \alpha) \text{ for all } x \text{ and all } i < \alpha. \) This implies that at any time \( t \) such that \( q_t < \alpha, \) the monopolist can sell to all remaining high types at price \( P(x_t, \alpha). \) Therefore, for all states \( (x, q) \) with \( q \in [0, \alpha), \) the monopolist’s profits are bounded below by

\[
L(x, q) = \sup_{r \in T} E \left[ e^{-r \tau} \left[ (\alpha - q)(P(x, \alpha) - x_r) + \Pi(x, \alpha) \right] \bigg| x_0 = x \right].
\]  \( (10) \)

At states \( (x, q) \) with \( q < \alpha \) the seller can pursue the following strategy: she can choose optimally the time \( \tau \) at which to sell to the remaining high type buyers at price \( P(x, \alpha), \) obtaining \( (\alpha - q)(P(x, \alpha) - x_r) \) from these sales plus a continuation payoff of \( \Pi(x, \alpha). \) By following this strategy, the seller earns profits equal to \( L(x, q). \)

**Lemma 3**  For every \( q \in [0, \alpha), \) there exists \( \underline{x}(q) \in (0, z_1) \) and \( \overline{x}(q) \in (z_1, z_2) \) such that \( \tau(q) = \inf \{ t : x_t \in [0, \underline{x}(q)] \cup [\overline{x}(q), z_2] \} \) solves \( (10). \) Moreover, \( \underline{x}(\cdot) \) and \( \overline{x}(\cdot) \) are continuous,
with \( \lim_{q \to \alpha} x(q) = \lim_{q \to \alpha} \overline{x}(q) = z_1 \).

**Proof.** See Appendix A.2.

To gain intuition behind the solution to (10), let \( g(x, q) = (\alpha - q)(P(x, \alpha) - x) + \Pi(x, \alpha) \), so that \( L(x, q) = \sup_{x \in T} E[e^{-rt} g(x, q) | x_0 = x] \). Since \( P(x, \alpha) \) has a convex kink at \( z_1 \) (see Figure 1) and since \( \Pi(x, \alpha) \in C^1 \), \( g(x, q) \) also has a convex kink at \( z_1 \). Therefore, when \( x \in (\underline{x}(q), \overline{x}(q)) \) the monopolist can obtain larger profits by delaying trade with high type buyers than by serving all of them at price \( P(x, \alpha) \) (see Figure 2). Intuitively, the monopolist cannot sell to all remaining high types at a price significantly larger than \( v_1 \) when \( x \in (\underline{x}(q), \overline{x}(q)) \), since high types expect prices to drop fast to \( v_1 \) after they all buy and leave the market. However, the monopolist has the option of selling to high types at a future point in time. The cutoffs \( \underline{x}(q) \) and \( \overline{x}(q) \) are such that the monopolist gets a larger payoff by waiting than by selling to all remaining high types immediately whenever \( x \in (\underline{x}(q), \overline{x}(q)) \). The solution to (10) also involves delaying when costs are above \( z_2 \): serving high types is too expensive when \( x > z_2 \), so in this case it is optimal to wait for costs to fall.

The proof of Lemma 3 shows that, for all \( x \in (\underline{x}(q), \overline{x}(q)) \), \( L(x, q) \) solves

\[
\frac{d^2}{dx^2} L(x, q) = \mu x L_x(x, q) + \frac{1}{2} \sigma^2 L_{xx}(x, q) .
\]

The general solution to (11) is \( L(x, q) = Ax^\lambda + Bx^\kappa \), where \( \lambda < 0 \) and \( \kappa > 1 \) are the roots of \( \frac{1}{2} \sigma^2 \lambda (\lambda - 1) + \mu \lambda = r \), and \( A \) and \( B \) are constants. There are four unknowns: \( A \), \( B \) and
the thresholds $\underline{x}(q)$ and $\overline{x}(q)$. The four equations that determine these unknowns are

$$L(x(q), q) = g(x(q), q), \quad L_x(x(q), q) = g_x(x(q), q), \quad (VM)$$

$$L_x(x(q), q) = g_x(x(q), q), \quad L_x(\overline{x}(q), q) = g_x(\overline{x}(q), q). \quad (SP)$$

The optimal stopping problem (10) is defined for all $q \in [0, \alpha)$: for each $q \in [0, \alpha)$ there are cutoffs $\underline{x}(q)$ and $\overline{x}(q)$ such that the solution to (10) involves stopping the first time $x_t \in [\underline{x}(q), \overline{x}(q)]$. Lemma 3 shows that $\underline{x}(\cdot)$ and $\overline{x}(\cdot)$ are continuous, with $\lim_{q \to \alpha^-} \underline{x}(q) = \lim_{q \to \alpha^-} \overline{x}(q) = z_1$. In words, the delay region $(\underline{x}(q), \overline{x}(q))$ shrinks as $q$ increases, and as $q$ converges to $\alpha$ it becomes optimal to stop when $x_t \leq z_2$. The intuition behind this result is as follows. The seller benefits by delaying trade with high types when $x \in (\underline{x}(q), \overline{x}(q))$, since this allows her to obtain more rents out of them. However, this delayed trade with high types comes at a cost to the seller, since it implies that she will be delaying trade with low types beyond what is optimal. Note that the gains from delaying trade decrease as there are fewer high types in the market, while the costs remain the same. As a result of this, the delay region $(\underline{x}(q), \overline{x}(q))$ shrinks as $q$ increases.

The following Theorem, which is the main result of this section, shows that the monopolist’s equilibrium profits are exactly equal to the lower bound $L(x, q)$.

**Theorem 1** There exists a unique equilibrium. In equilibrium, at every state $(x, q)$ with $q \in [0, \alpha)$ the monopolist’s profits are $L(x, q)$. Moreover, for all $t \geq 0$ with $q_t < \alpha$,

(i) if $x_t > z_2$, the monopolist doesn’t sell, so $dq_t = 0$,

(ii) if $x_t \in [\overline{x}(q_t), z_2]$, the monopolist sells to all remaining high type consumers at price $P(x_t, \alpha)$, so $dq_t = \alpha - q_t$,

(iii) if $x_t \leq \underline{x}(q_t)$, the monopolist sells to all remaining consumers (high and low types) at price $v_1$, so $dq_t = 1 - q_t$,

(iv) while $x_t \in (\underline{x}(q_t), \overline{x}(q_t))$, the monopolist gradually sells to high type consumers at price $P(x_t, q_t) = x_t - L_q(x_t, q_t)$, so $q_t$ is continuous and strictly increasing.

**Proof.** See Appendix A.3. $lacksquare$

Theorem 1 shows that the monopolist’s profits are equal to the lower bound $L(x, q)$ for every state $(x, q)$ with $q \in [0, \alpha)$. When $x_t \in [\overline{x}(q_t), z_2]$, the monopolist sells to all remaining high type buyers at price $P(x_t, \alpha)$, and then sells to low types when costs drop below $z_1$. 

13
When \( x_t \leq \bar{x}(q_t^-) \), the monopolist sells to both low and high type consumers at price \( v_1 \) and the market closes. When \( x_t > z_2 \), the monopolist waits for costs to decrease. Finally, when \( x_t \in (\bar{x}(q_t^-), \bar{x}(q_t^-)) \) it is never optimal for the monopolist to sell to all remaining high type buyers immediately: by doing this the monopolist earns \( g(x_t, q_{t^-}) < L(x_t, q_{t^-}) \). Instead, when costs are in this region the monopolist sells to high type consumers gradually over time, and market penetration increases continuously. Since \( \bar{x}(q) \) is decreasing in \( q \) and \( \bar{x}(q) \) is increasing in \( q \) (Lemma A6), the interval \((\bar{x}(q), \bar{x}(q))\) shrinks as \( q \) increases.

I now show how to determine the price that the monopolist charges and the rate at which she sells when \( x_t \in (\bar{x}(q_t), \bar{x}(q_t)) \). Suppose \( x_t \in (\bar{x}(q_t), \bar{x}(q_t)) \) and let \( \tau = \inf\{s > t : x_s \notin (\bar{x}(q_s), \bar{x}(q_s))\} \). By Theorem 1, \( q_s \) is continuous and strictly increasing for all \( s \in [t, \tau) \). At any \( s \in [t, \tau) \), the monopolist’s expected discounted profits (which by Theorem 1 are equal to \( L(x_s, q_s) \)) are given by

\[
L(x_s, q_s) = E\left[ \int_s^\tau e^{-r(u-s)} (P(x_u, q_u) - x_u) \, dq_u + e^{-r(\tau-s)} L(x_\tau, q_\tau) \, \mathcal{F}_s \right].
\]

By the Law of Iterated Expectations, the process

\[
Y_s = \int_0^s e^{-ru} (P(x_u, q_u) - x_u) \, dq_u + e^{-r \tau} L(x_s, q_s)
\]

is a continuous martingale in \( s \in [t, \tau) \). By the Martingale Representation Theorem (Karatzas and Shreve, 1998), there exists a progressively measurable process \( \beta \in \mathcal{L}^* \) such that \( dY_s = e^{-r \beta_s} dB_s \).\(^{10}\) Differentiating (12) with respect to \( s \) and using \( dY_s = e^{-r \beta_s} dB_s \) gives

\[
\begin{align*}
\frac{dY_s}{ds} &= e^{-\beta_s} (P(x_s, q_s) - x_s) \, ds + e^{-r \beta_s} L(x_s, q_s) \, ds + e^{-r \beta_s} dB_s,
\end{align*}
\]

\[
\begin{align*}
\frac{dL(x_s, q_s)}{ds} &= rL(x_s, q_s) \, ds + (P(x_s, q_s) - x_s) \, ds + \beta_s \, dB_s.
\end{align*}
\]

Since \( L(x, q) \in C^2 \) for all \( x \in (\bar{x}(q), \bar{x}(q)) \) (Lemma A5 in the Appendix), by Ito’s Lemma

\[
dL(x_s, q_s) = \left( \mu x_s L_x (x_s, q_s) + \frac{1}{2} \sigma^2 x_s^2 L_{xx} (x_s, q_s) \right) \, ds + L_q (x_s, q_s) \, dq_s + \sigma x L_x (x_s, q_s) \, dB_s.
\]

Combining these two equations, the profit function of the monopolist satisfies the following

\(^{10}\) A process \( \beta \) belongs to \( \mathcal{L}^* \) if \( E[\int_0^t \beta_s^2 \, ds] < \infty \) for all \( t \in [0, \infty) \).
Figure 3: Prices $P(x, q)$; $v_1 = \frac{1}{2}$, $v_2 = 1$, $\alpha = 0.7$, $\mu = -0.02$, $\sigma = 0.25$ and $r = 0.05$.

Bellman equation at all states $(x_s, q_s)$ with $x_s \in (\underline{x}(q_s), \overline{x}(q_s))$,

$$rL(x_s, q_s) = (P(x_s, q_s) - x_s) \frac{dq_s}{ds} + L_q(x_s, q_s) \frac{dq_s}{ds} + \mu x_s L_x(x_s, q_s) + \frac{1}{2} \sigma^2 x_s^2 L_{xx}(x_s, q_s).$$

(13)

The left-hand side of (13) is the monopolist’s flow payoff at state $(x_s, q_s)$, while the right-hand side shows the sources of this flow payoff. The term $(P(x_s, q_s) - x_s) \frac{dq_s}{ds}$ represents the flow payoffs that the monopolist gets from her sales, while the term $L_q(x_s, q_s) \frac{dq_s}{ds}$ represents the drop in the monopolist’s continuation payoff due to the fact that consumers are leaving the market at rate $dq_s/ds$. Finally, the term $\mu x_s L_x + \frac{1}{2} \sigma^2 x_s^2 L_{xx}$ gives the change in the seller’s payoff due to changes in costs. Comparing equations (13) and (11), it follows that

$$P(x_s, q_s) - x_s = -L_q(x_s, q_s),$$

(14)

for all $(x_s, q_s)$ such that $x_s \in (\underline{x}(q_s), \overline{x}(q_s))$. That is, the profit margin $P(x_s, q_s) - x_s$ that the monopolist earns must be equal to the cost $-L_q(x_s, q_s)$ that she incurs in terms of a lower continuation payoff. Equation (14) has the following interpretation. The monopolist sells at rate $dq_s > 0$ when $x_s \in (\underline{x}(q_s), \overline{x}(q_s))$. If $P(x_s, q_s) - x_s > -L_q(x_s, q_s)$, the monopolist could increase her profits by selling at a faster rate. Similarly, if $P(x_s, q_s) - x_s < -L_q(x_s, q_s)$ she would be better off not selling at all. Therefore, for $dq_s > 0$ to be optimal, equation (14) must hold for all $x_s \in (\underline{x}(q_s), \overline{x}(q_s))$. Figure 3 plots the price $P(x, q)$ that the seller charges when $x \in (\underline{x}(q), \overline{x}(q))$, for different values of $q$. 

\cite{Fuchs2010} derive a Bellman equation similar to (13) for their bargaining game.
Next, I pin down the rate $dq_t$ at which the monopolist sells when $x_t \in (\underline{x}(q_t), \overline{x}(q_t))$. Note that all high types must get the same payoff; otherwise, it would be profitable for a buyer getting a lower payoff to mimic the strategy of one who is getting a larger payoff. Therefore, prices must evolve in such a way that high types are indifferent between buying at any $s \in [t, \tau)$ (where $\tau = \inf \{s > t : x_s \notin (\underline{x}(q_s), \overline{x}(q_s))\}$). That is,

\[ v_2 - P(x_s, q_s) = E[e^{-r(u-s)}(v_2 - P(x_u, q_u)) | F_s] \Rightarrow \]

\[ e^{-rs}(v_2 - P(x_s, q_s)) = E[e^{-ru}(v_2 - P(x_u, q_u)) | F_s], \]  

(15)

for any $s, u \in [t, \tau)$, $s < u$. By the Law of Iterated Expectations, the process $M_s = E[e^{-ru}(v_2 - P(x_u, q_u)) | F_s]$ is a continuous martingale. By the Martingale Representation Theorem, there exists a progressively measurable process $\gamma \in \mathcal{L}^*$ such that $dM_s = e^{-rs}\gamma_s dB_s$. Differentiating (15) with respect to $s$ and using $dM_s = e^{-rs}\gamma_s dB_s$ gives

\[ dM_s = -re^{-rs}(v_2 - P(x_s, q_s)) ds - e^{-rs}dP(x_s, q_s) \Rightarrow \]

\[ dP(x_s, q_s) = -r(v_2 - P(x_s, q_s)) ds - \gamma_s dB_s. \]  

(16)

Equation (16) shows that (in expectation) prices must fall at rate $-r(v_2 - P(x_s, q_s))$ in order to keep high types indifferent between buying at any time $s \in [t, \tau)$. By equation (14), $P(x_s, q_s) = x_s - L_q(x_s, q_s)$ for all $s \in [t, \tau)$. The proof of Lemma 3 shows that $L(x, q) \in C^2$ for all $x \in (\underline{x}(q), \overline{x}(q))$, so $P(x, q) \in C^{2,1}$ for all $x \in (\underline{x}(q), \overline{x}(q))$. By Ito’s Lemma,

\[ dP(x_s, q_s) = \left(\mu x_s P_x(x_s, q_s) + \frac{1}{2} \sigma^2 x^2 P_{xx}(x_s, q_s)\right) ds + P_q(x_s, q_s) dq_s + P_x(x_s, q_s) \sigma x_s dB_s, \]

for all $s \in [t, \tau)$. Combining this expression with equation (16) and rearranging gives

\[ \frac{dq_s}{ds} = \dot{q}_s = \frac{-r(v_2 - P(x_s, q_s)) - \mu x_s P_x(x_s, q_s) - \frac{1}{2} \sigma^2 x^2 P_{xx}(x_s, q_s)}{P_q(x_s, q_s)}. \]

The proof of Lemma 3 shows that $L_q(x, q)$ solves $rL_q(x, q) = \mu x L_{qx}(x, q) + \frac{\sigma^2 x^2}{2} L_{qxx}(x, q)$ for $x \in (\underline{x}(q), \overline{x}(q))$ (see footnote 16 in Appendix A.2). Using this and equation (14) gives

\[ \dot{q}_s = \frac{-r(v_2 - x_s) + \mu x_s}{P_q(x_s, q_s)} = \frac{r(v_2 - x_s) + \mu x_s}{L_{qq}(x_s, q_s)} > 0, \]  

(17)

where the inequality follows since $L(x, q)$ is strictly convex in $q$ for all $x \in (\underline{x}(q), \overline{x}(q))$ (Lemma A6 in Appendix A.2) and $r(v_2 - x) + \mu x > 0$ for all $x < z_2$. Equation (17) gives
the rate at which the monopolist sells while \( x_s \in (x(q_s), x(q_s)) \).

**Remark 2** Theorem 1 establishes that the monopolist’s profits are equal to the lower bound \( L(x,q) \). This fact greatly simplifies the derivation of the equilibrium, since it allows me to pin down the strategies that players must use in order for the monopolist to earn profits equal to \( L(x,q) \). This simplification would not possible in a discrete time version of this game. The reason for this is that, in discrete time, the monopolist would be able to commit not to reduce prices during each time period; and this commitment ability would allow her to obtain profits strictly larger than the lower bound. It is this particular property of the continuous time model that makes the analysis simpler and cleaner, and that allows me to obtain a tractable characterization of the equilibrium dynamics in this environment with time-varying costs.\(^{12}\)

### 4.2 Features of the equilibrium

In this section I present the more salient features of the equilibrium. I start by discussing how the equilibrium outcome relates to the results on the Coase conjecture. Recall that in markets with time-invariant costs, the Coase conjecture predicts that the monopolist will post an opening price equal to the lowest valuation, and that all buyers will trade immediately at this price. The market outcome will therefore be efficient, and the monopolist will earn the same profits she would have earned if all buyers in the market had the lowest valuation.

With time-varying costs, selling to all consumers immediately is in general not efficient. By Lemma 1, efficiency requires the monopolist to serve the different types of buyers sequentially as costs decrease. On the other hand, the profits that a monopolist would earn if all buyers had the lowest valuation are \( V_1(x_0) = \sup_{\tau} E[e^{-\tau r}(v_1 - x_\tau)|x_0] \), since in this case the monopolist would sell to all consumers at a price equal to \( v_1 \). These observations suggest the following generalization of the Coase conjecture for markets with time-varying costs:

**Definition 2** An outcome is Coasian if (i) it is efficient, and (ii) the monopolist’s profits are equal to \( V_1(x_0) \).

By Definition 2, an outcome is Coasian if it is efficient and if the monopolist earns the same profits she would earn in a market in which all consumers have the lowest valuation.

\(^{12}\)Fuchs and Skrzypacz (2010) show that a similar property holds in their bargaining game with arrival of new traders: they show that, in the limit as the time period goes to zero, the seller’s profits are equal to her outside option of waiting for the arrival of new trader. This fact also greatly simplifies the equilibrium analysis in their setting.
Figure 4: Profits as a fraction of $\Pi^{FC}$; $\alpha = 0.7$, $v_1 = \frac{1}{2}$, $v_2 = 1$, $\mu = -0.02$, $\sigma = 0.2$ and $r = 0.05$.

$v_1$. Therefore, under a Coasian outcome the monopolist is unable to extract more rents from buyers with higher valuations than what she extracts from buyers with valuation $v_1$.

The equilibrium outcome fails to be Coasian when there are two types of consumers in the market. First, the monopolist is able to extract additional rents from those buyers with higher valuation, since time-varying costs endogenously provide commitment power. Indeed, $P(x, \alpha) - x > V_1(x)$ for all $x \in (z_1, z_2]$ (Lemma 2), so $L(x, 0) \geq \alpha (P(x, \alpha) - x) + (1 - \alpha) V_1(x) > V_1(x)$ for all $x \in (z_1, z_2]$. Second, the equilibrium is inefficient whenever $x_0 \in (x(0), \overline{x}(0))$. The monopolist sells to high type consumers gradually when costs initially lie within this range, but the efficient outcome is to serve them immediately. Note however that the equilibrium is efficient when costs initially lie either above $\overline{x}(0)$ or below $x(0)$.

A way to measure the size of the rents that the monopolist extracts from high type buyers is to compare the monopolist’s profits $L(x, 0)$ to the profits $\Pi^{FC}(x)$ she would earn if she could commit to a path of prices. One can show that $\Pi^{FC}(x) = \sup_r E[e^{-rT} \alpha (v_2 - x_T)] | x_0 = x]$ when $\alpha v_2 > v_1$. That is, under full commitment the monopolist would find it optimal to sell only to high types (at a price of $v_2$) when the share of high types is large. By Lemma 1, in this case the monopolist would serve high types the first time costs fall below $z_2$.13 High types would be willing to pay a price of $v_2$, since the monopolist can commit to keep prices above $v_2$ after they purchase. Figure 4 shows that the monopolist may obtain a substantial fraction of the full commitment profits when costs are time-varying.

13When $\alpha v_2 < v_1$, the monopolist’s full commitment strategy involves selling to high types the first time costs fall below $z_2$, and selling to low types the first time costs fall below some $z < z_1$. The price that the monopolist charges high types in this case leaves them indifferent between buying at that price or waiting until costs fall below $z$ and getting the good at a price equal to $v_1$. 

18
Another salient feature of the equilibrium is that the price that the monopolist charges at time $t > 0$ may depend upon the history of costs. To see this, suppose that $x_0 \in (\underline{x}(0), \overline{x}(0))$ and let $\tau = \inf \{t : x_t \notin (\underline{x}(q_t), \overline{x}(q_t))\}$. By equation (17), the rate $\dot{q}_s$ at which the monopolist sells at time $s \in [0, \tau)$ depends on the current cost $x_s$ and on the current level of market penetration $q_s$. Therefore, for all $t \in [0, \tau)$ the level of market penetration $q_t = \int_0^t \dot{q}_s ds$ depends upon the path of costs from time zero to $t$; and so the price $P(x_t, q_t)$ that the seller charges at time $t$ also depends upon the history of costs. Figure 5 plots the path of prices and the evolution of market penetration for a path of costs with $x_0 \in (\underline{x}(0), \overline{x}(0))$. Note that $x_\tau = \overline{x}(q_{\tau-})$ under this path of costs. Therefore, at time $\tau$ the monopolist sells to all remaining high type buyers at price $P(x_\tau, \alpha)$. Since $P(x, q)$ is increasing in $x$, at time $\tau$ a mass of consumers buys at a moment at which prices are increasing.

In settings in which costs are time-invariant, the literature on the Coase conjecture refers to the difference between the lowest consumer valuation and the monopolist’s cost as the gap. When costs don’t change over time, the price that the monopolist charges to high type buyers is increasing in the gap (since, by the Coase conjecture, the monopolist posts an opening price equal to the lowest valuation). In this paper’s setting, we can think of $v_1$ as measuring the “gap”. The next result shows that the price at which the monopolist can sell to all high type buyers may be increasing or decreasing in the gap when costs are time-varying.
**Proposition 1** For all $x > z_1$, $P(x, \alpha)$ is increasing in $v_1$ if and only if $v_1 \geq \frac{1}{1-\lambda}v_2 = z_2$.

Proposition 1 follows from differentiating $P(x, \alpha)$ in equation (9) with respect to $v_1$. The price at which all high type buyers are willing to buy is decreasing in $v_1$ for low values of $v_1$, and its increasing in $v_1$ for high values of $v_1$. The price $P(x, \alpha)$ depends on two quantities: the time $\tau_1$ at which the monopolist sells to low type consumers (i.e., the endogenous commitment power of the monopolist), and the price $v_1$ that the monopolist charges at time $\tau_1$. An increase in $v_1$ affects both quantities: it decreases the stopping time $\tau_1$ (thereby decreasing $P(x, \alpha)$) and it increases the price $v_1$ that the monopolist charges at $\tau_1$ (thereby increasing $P(x, \alpha)$). By Proposition 1, the second effect dominates when $v_1$ is large relative to $v_2$, while the first effect dominates when $v_1$ is small.

The next Proposition shows how $P(x, \alpha)$ depends on the drift and volatility of costs.

**Proposition 2** For all $x > z_1$, (a) $\partial P(x, \alpha)/\partial \sigma > 0$ if and only if $x < x^* = z_1e^{\frac{1}{1-\lambda}}$, and (b) $\partial P(x, \alpha)/\partial \mu > 0$ if and only if $x > x^*$.

Proposition 2 follows from differentiating equation (9) with respect to $\mu$ and $\sigma$. When costs are above $x^* = z_1e^{\frac{1}{1-\lambda}}$, the monopolist is able to charge a higher price to high type consumers in environments in which costs are more volatile. A change in $\sigma$ has two opposing effects on the price $P(x, \alpha)$. First, an increase in $\sigma$ lowers the threshold $z_1$ at which the monopolist starts selling to low type buyers, leading to an increase in $P(x, \alpha)$. Second, a higher $\sigma$ means that costs will (on average) reach $z_1$ faster, leading to a decrease in $P(x, \alpha)$. Proposition 2 shows that the second effect dominates when costs lie between $z_1$ and $x^*$, and that the first effect dominates when costs are above $x^*$. Proposition 2 also shows that an increase in $\mu$ has the opposite effect on $P(x, \alpha)$ than an increase in $\sigma$. The reason for this is that changes $\mu$ have the opposite effect on both the threshold $z_1$ and on the speed with which $x_t$ reaches $z_1$ than changes in $\sigma$.

The final result of this section characterizes the equilibrium outcome in the limit as the drift and volatility of costs converge to zero; i.e., as costs become time-invariant. For any $x \in [0, v_2]$, let $p(x)$ denote the lowest consumer valuation that is weakly larger than $x$; that is, $p(x) = v_1$ if $x \leq v_1$, while $p(x) = v_2$ if $x \in (v_1, v_2)$.

**Proposition 3** Suppose $x_0 \leq v_2$. Then, as $(\sigma, \mu) \to (0, 0)$ the monopolist sells at $t = 0$ to all consumers with valuation larger than $x_0$ at a price $p(x_0)$.

**Proof.** See Appendix A.4. ■
Proposition 3 shows that the equilibrium outcome of this model converges to the standard Coase conjecture outcome when the drift and volatility of costs converges to zero; i.e., when costs become time-invariant. For instance, if costs are initially below \( v_1 \), the monopolist’s opening price converges to the lowest valuation \( v_1 \) as \((\sigma, \mu) \to (0, 0)\), and all consumers trade immediately at this price.

5 Markets with \( n \) types of consumers

In this section, I show how the results in Section 4 generalize to settings in which the function \( f : [0, 1] \to [\underline{v}, \overline{v}] \) describing the valuations of the consumers takes \( n \) values \( v_1 < \ldots < v_n \) (with \( n > 2 \)). For \( k = 1, \ldots, n \), let \( \alpha_k = \max\{i \in [0, 1] : f(i) = v_k\} \) be the highest indexed consumer with valuation \( v_k \), and let \( \alpha_{n+1} = 0 \). Note then that \( f(i) = v_k \) for all \( i \in (\alpha_{k+1}, \alpha_k] \).

As a first step, note that at states \((x, q)\) with \( q \in [\alpha_3, 1) \) there are one or two types of buyers in the market: buyers with valuation \( v_1 \) and (if \( q < \alpha_2 \)) buyers with valuation \( v_2 \). Thus, for states \((x, q)\) with \( q \in [\alpha_3, 1) \) the equilibrium is the one derived in Section 4.

Consider next states \((x, q)\) with \( q \in [\alpha_4, \alpha_3) \). At such a state there are \( \alpha_3 - q \) buyers with valuation \( v_3 \) remaining in the market. By equation (5), the strategy \( P(x, \alpha_3) \) of consumer \( \alpha_3 \) (the highest indexed buyer with valuation \( v_3 \)) satisfies \( P(x, \alpha_3) = v_3 - E[e^{-\tau_2} (v_3 - P_2(x_{\tau_2}, q_{\tau_2}))|x_0 = x] \), where \( \tau_2 = \inf\{t : x_t \leq z_2\} \) is the time at which the monopolist starts selling to consumers with valuation \( v_2 \) when all consumers with valuation \( v_3 \) have left the market. The skimming property implies that \( P(x, i) \geq P(x, \alpha_3) \) for all \( i \leq \alpha_3 \), so the monopolist can sell to all buyers with valuation \( v_3 \) at price \( P(x, \alpha_3) \). Therefore, at states \((x, q)\) with \( q \in [\alpha_4, \alpha_3) \) the monopolist’s profits are bounded below by

\[
L(x, q) = \sup_{\tau \in T} E \left[ e^{-\tau} \left( (\alpha_3 - q) (P(x_\tau, \alpha_3) - x_\tau) + e^{-\tau} L(x_\tau, \alpha_3) \right) \bigg| x_0 = x \right].
\] (18)

By arguments similar to those in the proof of Lemma 3, the solution to (18) involves delaying when costs are either in an interval around \( z_1 \) or in an interval around \( z_2 \): in these regions, the seller obtains a larger payoff by waiting than by selling immediately to all buyers with valuation \( v_3 \) at price \( P(x, \alpha_3) \). The solution to (18) also involves delaying when \( x > z_3 \), since producing at this levels of cost is too expensive. As in the two types case, the seller’s profits are equal to \( L(x, q) \) for all states \((x, q)\) with \( q \in [\alpha_4, \alpha_3) \). When \( x_t < z_3 \) lies in the delay region of (18), the monopolist sells gradually to buyers with valuation \( v_3 \) at price \( x_t - L_q(x_t, q) \). When \( x_t \) lies in the stopping region of (18), the monopolist sells to all remaining buyers with valuation \( v_3 \) at price \( P(x_t, \alpha_3) \). After this sale, the states jumps to \((x_t, \alpha_3)\) and play
continuous as in the two types case. When \( x_t > z_3 \) the monopolist doesn’t sell.

Following the same steps as in the derivation of (18), I can extend the lower bound \( L(x, q) \) to all \( q \in [0, 1] \) in such a way that, for \( k = 1, \ldots, n \) and all \( q \in [\alpha_{k+1}, \alpha_k) \),

\[
L(x, q) = \sup_{\tau \in T} E \left[ e^{-r\tau} \left( (\alpha_k - q) (P(x, \alpha_k) - x_\tau) + e^{-r\tau} L(x, \alpha_k) \right) \right] \quad x_0 = x,
\]

(19)

where \( P(x, \alpha_k) \) is the price consumer \( \alpha_k \) (the highest indexed buyer with valuation \( v_k \)) is willing to pay, and \( L(x, \alpha_k) \) is the lower bound to the monopolist’s profits at state \( (x, \alpha_k) \).

**Theorem 2** There exists a unique equilibrium. In equilibrium, the monopolist’s profits are equal to \( L(x, q) \) at every state \( (x, q) \).

**Proof.** See Appendix A.5. ■

Theorem 2 states that the results in Theorem 1 generalize to the case in which \( f \) takes any finite number of values. In this setting, the seller’s equilibrium profits are also equal to the lower bound \( L(x, q) \). At states \( (x, q) \) with \( q \in [\alpha_{k+1}, \alpha_k) \), buyers with valuation \( v_{k+1} \) and higher have already left the market. At these states, the solution to (19) involves delaying when \( x \) is around \( z_1, z_2, \ldots, z_{k-1} \), and when \( x > z_k \). If \( x < z_k \) is in the delay region of the optimal stopping problem (19), the monopolist sells gradually to those buyers with valuation \( v_k \) (the highest valuation remaining in the market) at price \( P(x, q) = x - L_q(x, q) \). If \( x > z_k \), the seller waits until costs fall to \( z_k \), and at this point sells to all buyers with valuation \( v_k \) at price \( P(x, \alpha_k) \). Finally, if \( x \) lies in the stopping region of (19), the monopolist sells to all remaining buyers with valuation \( v_k \) at price \( P(x, \alpha_k) \), and the state moves to \( (x, \alpha_k) \).

The outcome in this setting also fails to be Coasian. First, the monopolist is able to extract additional rents from those buyers with higher valuations. Indeed, arguments similar to those in Lemma 2 imply that, for all \( k \geq 2 \), \( P(x, \alpha_k) - x > V_1(x) \) for all \( x \in (z_1, z_k] \). Since the monopolist can sell to all consumers with valuation \( v_k \) and higher at a price of \( P(x, \alpha_k) \), it follows that \( L(x, q) > (1-q)V_1(x) \) for all \( x \in (z_1, z_k] \) and \( q < \alpha_2 \). Intuitively, after selling to buyers with valuation \( v_k \) the seller can commit to keep prices high until costs fall below \( z_{k-1} \); and this commitment power allows her to extract rents from consumers with higher valuations. Second, the equilibrium also involves inefficiencies in the form of delayed trade: when \( x_0 < z_n \) lies in the delay region of (19), the efficient outcome is to serve all buyers with valuation \( v_n \) immediately, but the seller serves them gradually. In contrast, the outcome is efficient when \( x_0 \geq z_n \): in this case, the seller serves the different types of consumers at the surplus maximizing time.
6 Continuous types

In this section, I study markets in which the valuations of the consumers are described by a continuous and decreasing function \( h : [0, 1] \to [\underline{v}, \overline{v}] \), with \( \overline{v} > \underline{v} > 0 \). I study such markets by considering a sequence of models with step functions \( \{f^j\} \), with \( f^j : [0, 1] \to [\underline{v}, \overline{v}] \) taking finitely many values for all \( j \), such that \( \{f^j\} \to h \) (i.e., \( \sup_{i \in [0,1]} |f^j(i) - h(i)| \to 0 \) as \( j \to \infty \)).

Given such a sequence \( \{f^j\} \), let \( L^j(x, q) \) denote the monopolist’s profits at state \((x, q)\) in an environment in which the valuations of the consumers are described by \( f^j \). Let \( V(x) = \sup_{x_0} E[e^{-rT} (v - x_0) | x_0 = x] \) denote the profits that the monopolist would earn if all consumers in the market had the lowest valuation \( \underline{v} > 0 \).

**Theorem 3** Fix a sequence of step functions \( \{f^j\} \) such that \( \{f^j\} \to h \). Then, the equilibrium outcome becomes Coasian as \( j \to \infty \): (i) \( \lim_{j \to \infty} L^j(x, 0) = V(x) \) for all \( x \), and (ii) in the limit the monopolist serves the different consumers at the efficient time.

**Proof.** See Appendix A.6. ■

Theorem 3 shows that the market outcome becomes Coasian when types become continuous: the monopolist’s profits converge to what she would earn in a market in which all consumers have the lowest valuation \( \underline{v} \), and the market outcome becomes efficient.

To gain intuition behind this result, consider first a setting with two types of buyers: high types, with valuation \( \overline{v} \), and low types, with valuation \( \underline{v} \). After high types buy and leave the market, the monopolist can truthfully commit to keep high prices until costs fall below \( \overline{z} = \frac{-\lambda}{1-\lambda} \underline{v} \). High type buyers know that prices won’t fall to \( \underline{v} \) until \( x_t \) falls below \( \overline{z} \), so they are willing to pay higher prices when costs are above \( \overline{z} \). Consider next a setting with three types of buyers, with valuations \( \overline{v}, (\overline{v} + \underline{v})/2 \) and \( \underline{v} \). In this setting, after all consumers with valuation \( \overline{v} \) buy and leave the market, the monopolist can only commit to keep prices high until costs fall below \( \frac{-\lambda}{1-\lambda} \frac{\overline{v} + \underline{v}}{2} \), since at this point it becomes optimal for her to sell to buyers with intermediate valuation. This limits the price buyers with valuation \( \overline{v} \) are willing to pay, since they can now wait for costs to fall to \( \frac{-\lambda}{1-\lambda} \frac{\overline{v} + \underline{v}}{2} \) and get the good at a lower price.

More generally, the proof of Theorem 3 shows that the price consumers are willing to pay monotonically decreases as types become a continuum. In the limit as the gap between valuations becomes vanishingly small, the monopolist cannot extract additional rents from buyers with higher valuations, and her profits fall to what she would earn in a market in which all consumers have the lowest valuation \( \underline{v} \). Intuitively, the monopolist looses all commitment power when she faces a continuum of types, since in this case she always has an incentive to serve the next consumer arbitrarily soon after her last sale.
Theorem 3 and the results in Sections 4 and 5 suggest that, with time-varying costs, a durable goods monopolist will be able to obtain more profits in markets in which there is a clear segmentation between consumers. Examples include markets with a clear geographic segmentation, or in the case of intermediate durable goods, settings in which firms from different industries demand the monopolist’s good.

The equilibrium outcome is efficient with a continuum of types: the monopolist serves consumers with valuation \( v \in [\underline{v}, \bar{v}] \) the first time costs fall below the threshold \( z_v = \frac{1}{1-\lambda} v \). This implies that for all \( t \geq 0 \), consumers with valuation above \( \frac{1}{1-\lambda} \times \min_{s \leq t} x_t \) have bought and left the market. Therefore, the level of market penetration at each time \( t \) is entirely determined by the lowest level of costs up to \( t \), and market penetration only increases at points in time at which costs fall below their minimum historical level.

The proof of Theorem 3 shows that the price that the monopolist charges to consumers with valuation \( v \) is \( x_{r_v} + V(x_{r_v}) \), where \( r_v = \inf\{ t : x_t \leq \frac{1}{1-\lambda} v \} \). Let \( H \) denote the cumulative distribution function of valuations implied by \( h : [0, 1] \rightarrow [\underline{v}, \bar{v}] \).¹⁴ Since the monopolist obtains a margin of \( V(x_{r_v}) \) on buyers with valuation \( v \), her profits are

\[
E \left[ \int_{\underline{v}}^{v} e^{-r_t v} V(x_{r_v}) \, dH(v) \, | \, x_0 = x \right] = \int_{\underline{v}}^{v} E \left[ e^{-r_t \tau_v} V(x_{r_v}) \, | \, x_0 = x \right] dH(v) = V(x),
\]

where the last equality follows since \( E[e^{-r_t \tau_v} V(x_{r_v}) | x] = V(x) \) for all \( x \) and all \( v \in [\underline{v}, \bar{v}] \).¹⁵

The next result characterizes market outcomes in settings in which costs fall deterministically over time.

**Proposition 4** Suppose \( \mu < 0 \) and \( x_0 > \underline{v} \). Then, as \( \sigma \to 0 \) the monopolist charges price equal to marginal cost, and she serves consumers with valuation \( v \) the first time costs fall below \( v \).

Proposition 4 follows from the characterization of market outcomes with a continuum of types. By equation (7), \( \lim_{\sigma \to 0} V(x) \to 0 \) for all \( x > \underline{v} \). Therefore, the market outcome becomes competitive as volatility becomes negligible: the price \( V(x_t) + x_t \) at which the monopolist sells her good converges to marginal cost in the limit as \( \sigma \to 0 \), and the seller’s profits converge to zero. Moreover, \( z_v = \frac{1}{1-\lambda} v \to v \) as \( \sigma \to 0 \), so in the limit the monopolist serves buyers with valuation \( v \) the first time costs fall below \( v \).

---

¹⁴That is, \( H(v) = 1 - u(v) \) for all \( v \in [\underline{v}, \bar{v}] \), where \( u(v) = \sup\{ i \in [0, 1] : h(i) \geq v \} \).

¹⁵To see this, note first that \( E[e^{-r_t \tau_v} V(x_{r_v}) | x] = V(x) \) for \( x \leq z_v = \frac{1}{1-\lambda} v \). On the other hand, for \( x > z_v \), \( E[e^{-r_t \tau_v} V(x_{r_v}) | x] = E \left[ e^{-r_t \tau_v} \left( e^{-r_{x_{r_v}}} (\underline{v} - x_{r_v}) \right) \right] x \right\} V(x_{r_v}) | x] = E \left[ e^{-r_t \tau_v} (\underline{v} - x_{r_v}) | x \right] = V(x) \).
As noted in Section 4.1, in markets with time-varying costs we can think of the lowest valuation \( v \) as measuring the “gap”. The final result of this section characterizes market outcomes in the limit as the gap converges to zero.

**Proposition 5** As \( v \to 0 \), the monopolist charges price equal to marginal cost.

Proposition 5 also follows from the characterization of the market outcome with a continuum of types. To see this, note that \( V(x) = (v - z_v)(x/z_v) \to 0 \) as \( v \to 0 \). Therefore, the market outcome also becomes competitive when the gap goes to zero: in the limit as \( v \to 0 \), the monopolist charges a price equal to marginal cost and earns zero profits.

Finally, the assumption that the function \( h : [0, 1] \to [v, \bar{v}] \) is continuous implies that the cdf of consumer valuations \( H(v) \) has convex support. This assumption is crucial for Theorem 3 to hold. To see this, suppose that the function \( h \) has a single discontinuity point at \( j \in (0, 1) \), with \( h(j) > h(j^+) = \lim_{i \downarrow j} h(i) \). In such a setting, after selling to all consumers \( i \leq j \) the monopolist is be able to commit not to reduce her price until costs fall below \( \frac{1}{1-\lambda} h(j^+) \); and this commitment ability would allow the monopolist to extract rents from buyers with valuation larger than \( h(j) \) when costs are above \( \frac{1}{1-\lambda} h(j^+) \). This ability of the monopolist to extract rents from buyers with valuation larger than \( h(j) \) would create a wedge between profit maximization and efficiency, just as in the two types case of Section 4.

7 Conclusion

This paper studies the problem of a durable goods monopolist with uncertain and time-varying costs. In this setting, the market outcome fails to be Coasian when the distribution of valuations is discrete, since time-varying costs endogenously provide commitment power to the monopolist. On the other hand, with a continuum of types the market outcome is Coasian: the monopolist is unable to extract rents, and the outcome is efficient.

The continuous time methods used in this paper lead to a tractable characterization of the equilibrium. The model delivers a variety of predictions about how prices and profits relate to the different features of the environment. The paper uses the idea of building intuitive restrictions into the definition of equilibrium to study the continuous time game. I follow a similar approach in Ortner (2011), where I study a continuous time bilateral bargaining game in which the players’ bargaining power varies stochastically over time.

Finally, throughout the paper I assumed that costs follow a diffusion process. This assumption allows me to obtain a clear distinction between markets depending on the distribution of consumer valuations: under this environment, time-varying provide commitment
power to the monopolist only when there are gaps in the distribution of valuations. An avenue for future research is to explore alternative ways through which time-varying costs may provide commitment power. One such way is by allowing for other types of cost processes. To see this, suppose that costs follow a two state Markov chain, taking values $x_L > 0$ and $x_H > x_L$. Under this process, there exists $v^* > 0$ such that it is efficient to serve only those consumers with valuation above $v^*$ when costs are $x_H$. Assume further that there is a continuum of valuations $[v, \overline{v}]$, with $v^* \in (\overline{v}, \underline{v})$. In this setting, after selling to all consumers with valuation above $v^*$ the monopolist can commit not to cut her price until costs fall to $x_L$; and this commitment power allows her to extract rents from consumers with valuation above $v^*$ when costs are $x_H$. Moreover, there may be inefficient delay in equilibrium: if costs are initially $x_L$, the monopolist may obtain larger profits by waiting until costs are $x_H$ than by selling to everyone immediately at price $\underline{v}$, since this would allow her to charge consumers with valuation above $v^*$ a higher price. In this case, the monopolist would sell to consumers with valuation above $v^*$ gradually when costs are initially low. The market outcome thus fails to be Coasian in this setting, even with a continuum of types.
A Appendix

A.1 Proofs of Lemmas 1 and 2

Fix $y_2 > y_1 > 0$ and let $\tau_y = \inf\{ t : x_t \notin (y_1, y_2) \}$. Let $\tau_{y_1} = \inf\{ t : x_t \leq y_1 \}$.

**Lemma A1** Let $g$ be a bounded function, and let $W$ be the solution to

$$rW (x) = \mu x W' (x) + \frac{1}{2} \sigma^2 x^2 W'' (x),$$

(A.1)

with $W (y_1) = g (y_1)$ and $W (y_2) = g (y_2)$. Then, $W (x) = E[e^{-rt} g (x_t)] \mid x_0 = x]$ for all $x \in (y_1, y_2)$.

**Proof.** Let $W$ satisfy (A.1) with $W (y_1) = g (y_1)$ and $W (y_2) = g (y_2)$. The general solution to (A.1) is $W (x) = Ax^\lambda + Bx^\kappa$, where $\lambda < 0$ and $\kappa > 1$ are the roots of $\frac{1}{2} \sigma^2 \lambda (\lambda - 1) + \mu \lambda = r$, and where $A$ and $B$ are constants determined by the boundary conditions $W (y_1) = g (y_1)$ and $W (y_2) = g (y_2)$:

$$A = \frac{g (y_2) y_2^\kappa - g (y_1) y_1^\kappa}{y_1^\kappa y_2^\kappa - y_1^\kappa y_2^\kappa}, \quad B = -\frac{g (y_2) y_1^\kappa - g (y_1) y_2^\kappa}{y_1^\kappa y_2^\kappa - y_1^\kappa y_2^\kappa} \tag{A.2}$$

Let $f (x, t) = e^{-rt} W (x)$. By Itô’s Lemma, for $x_t \in (y_1, y_2)$,

$$df (x_t, t) = e^{-rt} \left( -rW (x_t) + \mu x W' (x_t) + \frac{1}{2} \sigma^2 x^2 W'' (x_t) \right) dt + e^{-rt} \sigma x W' (x_t) dB_t$$

$$= e^{-rt} \sigma x W' (x_t) dB_t,$$

where the second equality follows from the fact that $W$ solves (A.1). Then,

$$E\left[ e^{-rt} g (x_{\tau_y}) \mid x_0 = x \right] = E\left[ f (x_{\tau_y}, \tau_y) \mid x_0 = x \right] = f (x, 0) + E \left[ \int_0^\tau df (x_t, t) \mid x_0 = x \right]$$

$$= W (x) + E \left[ \int_0^\tau e^{-rt} \sigma x W' (x_t) dB_t \mid x_0 = x \right] = W (x),$$

since $\int_0^\tau e^{-rt} \sigma x W' (x_t) dB_t$ is a Martingale with expectation zero. \(\blacksquare\)

**Corollary A1** Let $g$ be a bounded function, and let $w$ be a solution to (A.1) with $w (y_1) = g (y_1)$ and $\lim_{x \to \infty} w (x) = 0$. Then, $w (x) = E[e^{-rt} g (x_{\tau_{y_1}}) \mid x_0 = x]$ for all $x > y_1$. Moreover, $w (x) = g (y_1) (x/y_1)^{\lambda}$ for all $x > y_1$.

**Proof.** Since $w$ solves (A.1), it follows that $w (x) = Cx^\lambda + Dx^\kappa$. The conditions $w (y_1) = g (y_1)$ and $\lim_{x \to \infty} w (x) = 0$ imply $D = 0$ and $C = g (y_1) (1/y_1)^{\lambda}$, so $w (x) = g (y_1) (x/y_1)^{\lambda}$. Next, note that for all $x_0 > y_1, \tau_y \to \tau_{y_1}$ as $y_2 \to \infty$. Then, by monotone convergence,

$$W (x) = E\left[ e^{-rt} g (x_{\tau_y}) \mid x_0 = x \right] \to E\left[ e^{-rt} g (x_{\tau_{y_1}}) \mid x_0 = x \right] \text{ as } y_2 \to \infty.$$
By Lemma A1, \( W(x) = Ax^\lambda + Bx^\kappa \) for \( x \in (y_1, y_2) \), with \( A \) and \( B \) satisfying (A.2). Since \( \lim_{y_2 \to \infty} B = 0 \) and \( \lim_{y_1 \to 0} A = g(y_1)/y_1^\lambda \), it follows that, for all \( x > y_1 \), \( E[e^{-\tau_{y_1}} g(x_{\tau_{y_1}}) | x_0 = x] = \lim_{y_2 \to \infty} W(x) = w(x) \). ■

**Proof of Lemma 1.** Let \( V_k(\cdot) \) be as in the statement of the Lemma. Note that \( V_k \) is twice differentiable with a continuous first derivative. One can show that \( V_k(x) > v_k - x \) for \( x > z_k \), so \( V_k(x) \geq v_k - x \) for all \( x \geq 0 \). Note also that \( V_k(\cdot) \) solves (A.1) for all \( x > z_k \), with \( V_k(z_k) = v_k - z_k \) and \( \lim_{x \to \infty} V_k(x) = 0 \). By Corollary A1, \( V_k(x) = E[e^{-\tau_k}(v_k - x_{\tau_k}) | x_0 = x] \). Moreover, \( (v_k - x) = rV_k(x) > \mu x V'_k(x) + \frac{1}{2} \sigma^2 x^2 V''_k(x) = -\mu x \), for all \( x \leq z_k \). Therefore, \( V_k \) is twice differentiable with a continuous first derivative, and satisfies

\[
-rV_k(x) + \mu x V'_k(x) + \frac{1}{2} \sigma^2 x^2 V''_k(x) \leq 0, \text{ with equality on } (z_k, \infty).
\]

Then, by standard verification theorems (e.g., Theorem 3.17 in Shiryaev, 2008) \( V_k(\cdot) \) is the solution to (6). ■

**Remark A1** Since \( V_k \) is a solution to the optimal stopping problem (6), then \( e^{-\tau t}V_k(x_t) \) is superharmonic; i.e., \( V_k(x) \geq E[e^{-\tau t}V_k(x_t) | x_0 = x] \) for any stopping time \( \tau \) (e.g., Theorem 10.1.9 in Oksendal, 2008). I will use this property repeatedly in what follows.

**Proof of Lemma 2.** Equation (9) follows from Corollary A1. Moreover, for all \( x \in (z_1, z_2) \),

\[
P(x, \alpha) - x - V_1(x) = v_2 - x - E \left[ e^{-\tau_{y_1}} (v_2 - v_1) | x_0 = x \right] - E \left[ e^{-\tau_{y_1}} (v_1 - x_{\tau_1}) | x_0 = x \right]
\]

\[= v_2 - x - E \left[ e^{-\tau_{y_1}} (v_2 - x_{\tau_1}) | x_0 = x \right] > 0
\]

since by Lemma 1, \( v_2 - x = V_2(x) > E[e^{-\tau_{y_1}}(v_2 - x_{\tau_1}) | x_0 = x] \) for all \( x \in (z_1, z_2) \). ■

### A.2 Proof of Lemma 3

The proof of Lemma 3 is organized as follows. Lemmas A2 and A3 give properties of solutions to equation (A.1). Lemma A4 uses these properties to characterize the solution to the optimal stopping problem (10). Finally, Lemmas A5 and A6 prove properties of the solution to (10).

**Lemma A2** Let \( U \) and \( \tilde{U} \) be two solutions to (A.1). If \( \tilde{U}(y) > U(y) \) for some \( y > 0 \), then \( \tilde{U}'(x) > U'(x) \) for all \( x > y \), and so \( \tilde{U}(x) > U(x) \) for all \( x > y \). Similarly, if \( \tilde{U}(y) \leq U(y) \) and \( \tilde{U}'(y) > U'(y) \) for some \( y > 0 \), then \( \tilde{U}'(x) > U'(x) \) for all \( x < y \), and so \( \tilde{U}(x) < U(x) \) for all \( x < y \).

**Proof.** I prove the first statement of the Lemma. The proof of the second statement is symmetric and omitted. Suppose the claim is not true, and let \( y_1 > y \) be the smallest point with \( U'(y_1) = \tilde{U}'(y_1) \). Therefore, \( \tilde{U}'(x) > U'(x) \) for all \( x \in [y, y_1) \), so \( \tilde{U}(y_1) > U(y_1) \). Since \( U \) and \( \tilde{U} \) solve (A1), then

\[
\tilde{U}''(y_1) = \frac{2(r\tilde{U}'(y_1) - \mu y_1 \tilde{U}'(y_1))}{\sigma^2 y_1^2} > \frac{2(rU'(y_1) - \mu y_1 U'(y_1))}{\sigma^2 y_1^2} = U''(y_1).
\]
But this implies that \( U'(y_1 - \varepsilon) > \tilde{U}'(y_1 - \varepsilon) \) for \( \varepsilon > 0 \) small enough, a contradiction. □

**Lemma A3** Fix \( q \in [0, \alpha) \) and \( y \in (0, z_1) \), and let \( U_y(x) \) be the solution to (A.1) with \( U_y(y) = (1 - q)(v_1 - y) \) and \( U'_y(y) = -(1 - q) \). Then, \( U_y(x) \) is strictly convex for all \( x > 0 \). Moreover, if \( y < y' < z_1 \), then \( U_y(x) > U_{y'}(x) \) for all \( x \geq y' \).

**Proof.** Since \( U_y(\cdot) \) solves (A.1), it follows that \( U_y(x) = Ax^\lambda + Bx^\kappa \). The constants \( A \) and \( B \) are determined by the conditions \( U_y(y) = (1 - q)(v_1 - y) \) and \( U'_y(y) = -(1 - q) \):

\[
A = y^{-\lambda}(1-q)\frac{\kappa(v_1 - y) + y}{\kappa - \lambda} > 0 \quad \text{and} \quad B = y^{-\kappa}(1-q)\frac{-(v_1 - y)\lambda - y}{\kappa - \lambda} > 0,
\]

where the second inequality follows from the fact that \( y < z_1 = -v_1\lambda/(1-\lambda) \). Thus, \( U''_y(x) = \lambda(\lambda - 2)Ax^{\lambda-2} + \kappa(\kappa - 1)Bx^{\kappa-2} > 0 \) for all \( x > 0 \) (since \( \kappa > 1 \)). Finally, let \( y < y' < z_1 \). Since \( U_y(\cdot) \) is strictly convex, it follows that \( U_y(y') > (1-q)(v_1 - y') = U_{y'}(y') \) and \( U'_y(y') > -(1-q) = U'_{y'}(y') \). Hence, by Lemma A2 \( U_y(x) > U_{y'}(x) \) for all \( x \geq y' \). □

For \( q \in [0, \alpha) \) and \( x > 0 \), let \( g(x,q) = (\alpha - q) (P(x, \alpha) - x) + \Pi(x, \alpha) \). Thus, \( L(x,q) = \sup_x E[e^{-\tau}g(x,q)|x_0 = x] \). Note that \( g(x,q) = (1-q)(v_1 - y) \) for all \( x \leq z_1 \).

**Lemma A4** For all \( q \in [0, \alpha) \), there exists \( \underline{x}(q) \in (0, z_1) \) and \( \overline{x}(q) \in (z_1, z_2) \) such that \( \tau(q) = \inf\{t: x_t \in [\underline{x}(q), \overline{x}(q)] \} \) solves (10). Moreover,

(i) for all \( x \in (\underline{x}(q), \overline{x}(q)) \cup (z_2, \infty) \), \( L(x,q) \) solves (A.1), with \( \lim_{x \to \infty} L(x,q) = 0 \).

(ii) for all \( x \leq \underline{x}(q) \) and all \( x \in [\overline{x}(q), z_2] \), \( L(x,q) = g(x,q) \).

(iii) the cutoffs \( \underline{x}(q) \) and \( \overline{x}(q) \) are such that

\[
L(\underline{x}(q), q) = g(\underline{x}(q), q), \quad L(\overline{x}(q), q) = g(\overline{x}(q), q), \quad L_x(\underline{x}(q), q) = g_x(\underline{x}(q), q), \quad L_x(\overline{x}(q), q) = g_x(\overline{x}(q), q).
\]

**Proof.** First I show that there exists a function \( G(x,q) \) satisfying conditions (i)-(iii). Then I show that \( G(x,q) = \sup_x E[e^{-\tau}g(x,q)|x_0 = x] = L(x,q) \). I start by showing that there exists a function \( G(x,q) \) and unique cutoffs \( \underline{x}(q) \) and \( \overline{x}(q) \) such that \( G(x,q) \) solves (A1) on \( (\underline{x}(q), \overline{x}(q)) \) and satisfies (iii). To see this, consider solutions \( U \) to (A.1) with \( U(y) = g(y,q) = (1-q)(v_1 - y) \) and \( U'(y) = g_x(y,q) = -(1-q) \) for some \( y < z_1 \). By Lemma A3, such solutions are strictly convex. Since solutions to (A.1) are continuous in initial conditions, then the solutions I’m considering are continuous in \( y \). If \( y \) is small enough, then \( U(x) \) will remain above \( g(x,q) \) for all \( x > y \). On the other hand, if \( y \) is close to \( z_1 \) then \( U \) will cross \( g(x,q) \) at some \( \overline{x} > z_1 \) (see solutions I-IV in Figure A1). By Lemma A3, the point \( \overline{x} \) moves to the right as \( y \) decreases. Let \( \underline{x}(q) \) be the smallest \( y \) such that \( U \) reaches \( g(x,q) \) at some \( \overline{x} > y \). Since a solution with \( y < \underline{x}(q) \) never reaches \( g(x,q) \), it follows that \( U(x) \geq g(x,q) \) for all \( x \). Thus, \( U \) is tangent to \( g(x,q) \) at \( \overline{x}(q) \), so \( U'(\overline{x}(q)) = g_x(\overline{x}(q), q) \) (solution III in Figure A1). Let \( G(x,q) = U(x) \) for \( x \in [\underline{x}(q), \overline{x}(q)] \). By construction, it must be that \( \underline{x}(q) \in (0, z_1) \) and that \( \overline{x}(q) > z_1 \). One can also show that \( \overline{x}(q) < z_2 \).

Next, for all \( x > z_2 \), let \( G(x,q) \) be the solution to (A.1) with \( \lim_{x \to \infty} G(x,q) = 0 \) and \( G(z_2,q) = g(z_2,q) \). By Corollary A1, \( G(x,q) = E[e^{-\tau}g(x,q)|x_0 = x] = g(z_2,q)(x/z_2)^\lambda \).
for all $x > z_2$. For future reference, note that $G_x(z_2, q) = g_x(z_2, q)$. Also, one can check that $g(x, q) < G(x, q)$ for all $x > z_2$. Finally, for all $x \leq \bar{x}(q)$ and for all $x \in [\underline{x}(q), z_2]$, let $G(x, q) = g(x, q)$. By construction, $G(x, q)$ satisfies (i)-(iii).

I now show that $G(x, q) = L(x, q) = \sup_{x} E[e^{-\beta t} g(x, q)| x_0 = x]$. By construction, $G(x, q) \geq g(x, q)$ for all $x \geq 0$. Moreover, $G(x, q)$ is twice differentiable in $x$, with a continuous first derivative. Finally, the function $G(x, q)$ satisfies:

$$-rG(x, q) + \mu x G_{x}(x, q) + \frac{1}{2} \sigma^2 x^2 G_{xx}(x, q) \leq 0,$$

with equality on $(\bar{x}(q), \underline{x}(q)) \cup (z_2, \infty)$.

Indeed, $G(x, q)$ satisfies (A.3) with equality on $(\bar{x}(q), \underline{x}(q)) \cup (z_2, \infty)$ since it solves (A.1) in this region. One can also check that $rG(x, q) \geq \mu x G_{x}(x, q) + \frac{1}{2} \sigma^2 x^2 G_{xx}(x, q)$ for all $x \in [0, \bar{x}(q)] \cup [\underline{x}(q), z_2]$. Therefore, by standard verification theorems (e.g., Theorem 3.17 in Shiryaev, 2008), $G(x, q) = \sup_{x} E[e^{-\beta t} g(x, q)| x_0 = x] = L(x, q)$. By Lemma A1 and Corollary A1, $L(x, q) = E[e^{-\beta t} g(x, q)| x_0 = x]$, so $\tau(q)$ solves (10).

\textbf{Lemma A5} \textit{$L(x, q) \in C^2$ for all $x \in (\underline{x}(q), \bar{x}(q))$ and all $q \in [0, \alpha)$. Moreover, $\underline{x}(q)$ and $\bar{x}(q)$ are $C^2$, with $\lim_{x \to \alpha} \underline{x}(q) = \lim_{x \to \alpha} \bar{x}(q) = z_1$.}

\textbf{Proof.} By Lemma A4, $L(x, q) = A(q) x^\lambda + B(q) x^\kappa$ for all $x \in (\underline{x}(q), \bar{x}(q))$, where $A(q), B(q), \underline{x}(q)$ and $\bar{x}(q)$ are determined by the system of equations (VM) + (SP). Denote this system of equations by $F(\underline{x}(q), \bar{x}(q), A(q), B(q)) = 0$. One can check that $F \in C^2$ and its Jacobian at $(\underline{x}(q), \bar{x}(q), A(q), B(q))$ has a non-zero determinant. By the Implicit Function Theorem, the functions $A(q), B(q), \underline{x}(q)$ and $\bar{x}(q)$ are all $C^2$ with respect to $q$ (e.g., de la Fuente, 2000, pages 210-211). Since $L(x, q) = A(q) x^\lambda + B(q) x^\kappa$ for all $x \in (\underline{x}(q), \bar{x}(q))$, it follows that $L(x, q) \in C^2$ for all $x \in (\underline{x}(q), \bar{x}(q))$.\textsuperscript{16}

\textsuperscript{16}This also implies that $L_q(x, q) = A'(q) x^\lambda + B'(q) x^\kappa$ for all $x \in (\underline{x}(q), \bar{x}(q))$. Thus, $L_q(x, q)$ solves (A.1) for all $x \in (\underline{x}(q), \bar{x}(q))$.  

30
Next, I show that \( \lim_{q \to \alpha} \underline{x}(q) = \lim_{q \to \alpha} \bar{x}(q) = z_1 \). Let \( \underline{x} = \lim_{q \to \alpha} \underline{x}(q) \) and \( \bar{x} = \lim_{q \to \alpha} \bar{x}(q) \). Since \( \underline{x}(q) < z_1 \) and \( \bar{x}(q) > z_1 \) for all \( q \) (Lemma A4), it follows that \( \underline{x} \leq z_1 \leq \bar{x} \).

Let \( \tau = \inf \{ t : x_t \in [0, x] \cup [\bar{x}, z_2] \} \), so \( \tau(q_n) \to \tau \) for every sequence \( \{q_n\} \to \alpha \). Note that \( L(x, q) \geq g(x, q) \geq \Pi(x, \alpha) = (1 - \alpha) V_1(x) \) for all \( q \leq \alpha \), so \( \lim_{q \to \alpha} L(x, q) \geq (1 - \alpha) V_1(x) \).

Let \( \{q_n\} \to \alpha \). Since \( \lim_{q \to \alpha} g(x, q) = (1 - \alpha) V_1(x) \), by Dominated Convergence

\[
L(x, q_n) = E \left[ e^{-r \tau(q_n)} g(x_{\tau(q_n)}; q_n) \middle| x_0 = x \right] \to_{n \to \infty} E \left[ e^{-r \tau} (1 - \alpha) V_1(x) \middle| x_0 = x \right]
\]

Suppose by contradiction that \( x < z_1 \). Then, for \( x \in (\underline{x}, \bar{x}) \),

\[
E \left[ e^{-r \tau} V_1(x) \middle| x_0 = x \right] = \Pr(x_{\tau} = \underline{x}) E \left[ e^{-r \tau} (v_1 - \underline{x}) \middle| x_0 = x \right] + \Pr(x_{\tau} = \bar{x}) E \left[ e^{-r \tau} V_1(\bar{x}) \middle| x_0 = x \right] < V_1(x),
\]

where the inequality follows from Remark A1 and the fact that \( E[e^{-r \tau}(v_1 - \underline{x})] < V_1(x) = E[e^{-r \tau}(v_1 - z_1)] \) (by Lemma 1). Hence, \( (1 - \alpha) E[e^{-r \tau} V_1(x) \middle| x_0 = x] < (1 - \alpha) V_1(x) \), which contradicts \( \lim_{q \to \alpha} L(x, q) \geq (1 - \alpha) V_1(x) \). Thus, it must be that \( x = z_1 \).

Suppose next that \( \bar{x} > z_1 \). Let \( W(x) = E[e^{-r \tau}(P(x_{\tau}, \alpha) - x_{\tau}) \middle| x_0 = x] \). Since \( x = z_1 \), it follows that \( \tau = \inf \{ t : x_t \in [0, z_1] \cup [\bar{x}, z_2] \} \). Let \( Y_t = e^{-r t}(P(x_t, \alpha) - x_t) \). By Ito’s Lemma, for all \( x_t \in (z_1, \bar{x}) \),

\[
dY_t = e^{-r t} \left( -r(P(x_t, \alpha) - x_t) + \mu x_t (P_x(x_t, \alpha) - 1) + \frac{\sigma^2 x_t^2}{2} P_{xx}(x_t, \alpha) \right) dt + \sigma x_t P_x(x_t, \alpha) dB_t
\]

where the second equality follows since equation (9) implies that \( r P(x, \alpha) = r v_2 + \mu x P_x(x, \alpha) + \frac{\sigma^2 x^2}{2} P_{xx}(x, \alpha) \) for all \( x > z_1 \). Therefore, for \( x \in (z_1, \bar{x}) \),

\[
W(x) = E[Y_{\tau} \middle| x_0 = x] = Y_0 + E \left[ \int_{0}^{\tau} e^{-r t} (-r (v_2 - x_t) - \mu x_t) dt \middle| x_0 = x \right].
\]

One can check that \( -r (v_2 - x) < \mu x \) for all \( x < \bar{x} < z_2 \), so \( W(x) < Y_0 = P(x, \alpha) - x \).

For each \( q \in [0, \alpha] \), let \( W(x, q) = E[e^{-r \tau(q)} (P(x_{\tau(q)}, \alpha) - x_{\tau(q)}) \middle| x_0 = x] \). Pick a sequence \( \{q_n\} \to \alpha \), so that \( \tau(q_n) \to \tau \) as \( n \to \infty \). By dominated convergence, \( W(x, q_n) \to W(x) \) as \( n \to \infty \). Fix \( x \in (z_1, \bar{x}) \). Since \( W(x) < P(x, \alpha) - x \), there exists \( N \) such that \( W(x, q_n) < P(x, \alpha) - x \) for all \( n > N \). On the other hand, \( E[e^{-r \tau(q_n)} V_1(x_{\tau(q_n)}) \middle| x_0 = x] \leq V_1(x) \) for all \( x \) and all \( n \) (see Remark A1). Therefore, for \( n > N \)

\[
L(x, q_n) = E \left[ e^{-r \tau(q_n)} \left( (\alpha - q_n)(P(x_{\tau(q_n)}, \alpha) - x_{\tau(q_n)}) + (1 - \alpha) V_1(x_{\tau(q_n)}) \right) \middle| x_0 = x \right]
\]

\[
< (\alpha - q_n)(P(x, \alpha) - x) + (1 - \alpha) V_1(x) = g(x, q_n),
\]

which contradicts the fact that \( L(x, q_n) = \sup_{\tau} E \left[ e^{-r \tau} g(x_{\tau}, q_n) \middle| x_0 = x \right] \). Thus, \( \bar{x} = z_1 \).

**Proof of Lemma 3.** Follows directly from Lemmas A4 and A5. ■
Lemma A6 \( L(x,q) \) is strictly convex in \( q \) for all \( x \in (\underline{x}(q), \overline{x}(q)) \). Moreover, \( \underline{x}'(q) > 0 \) and \( \overline{x}'(q) < 0 \).

**Proof.** I first show that \( \underline{x}'(q) > 0 \) and \( \overline{x}'(q) < 0 \). For \( q \in [0, \alpha) \), let \( W(x,q) = E[\epsilon q(x - \alpha)]P(x - \alpha)\] and \( U(x,q) = E[\epsilon q(x - \alpha)]V(x - \alpha)\] so \( L(x,q) = (\alpha - q)W(x,q) + (1 - \alpha)U(x,q) \). By Lemma A1, \( U(x,q) \) solves (A.1) for all \( x \in (\underline{x}(q), \overline{x}(q)) \) with \( U(x,q) = v_1 - \underline{x} = V_1(\overline{x}(q)) \) and \( U(\overline{x}(q),q) = (v_1 - z_1)(\overline{x} - z_1) = V_1(\overline{x}(q)) \). I now show that \( U_x(x,q) < V_1'(\overline{x}(q)) = -1 \) and \( U_x(x,q) > V_1'(\overline{x}(q)) \). To see this, note that \( V_1(x) \) also solves (A.1) for \( x \geq z_1 \), with \( V_1(z_1) = v_1 - z_1 \) and \( V'(z_1) = -1 \). Suppose by contradiction that \( U_x(\overline{x}(q),q) \geq -1 \). By Lemmas A2 and A3, \( U'(x) > -1 \) and \( U(x) > v_1 - x \) for all \( x > \overline{x}(q) \). Lemma A2 then implies that \( U(x,q) > V_1(x) \) for all \( x > \overline{x}(q) \), a contradiction to the fact that \( U(\overline{x}(q), q) = V_1(\overline{x}(q)) \). Hence, \( U_x(\overline{x}(q),q) < -1 \). A symmetric argument establishes that \( U_x(\overline{x}(q),q) > V_1'(\overline{x}(q)) \). Since \( L_x(x,q) = g_x(x,q) = -(1-q) \) and \( L_x(\overline{x}(q),q) = \overline{g_x(\overline{x}(q),q)} = (\alpha - q)(P_x(\overline{x}(q),q) - 1) + (1-\alpha)V_1'(\overline{x}(q)) \), it follows that \( W_x(x,q) \geq -1 \) and \( W_x(\overline{x}(q),q) < V_1'(\overline{x}(q)) \).

Fix \( q' < q < \alpha \). Let \( F(x) = (\alpha - q')(W(x,q) + (1 - \alpha)U(x,q)) \). Since \( W(x,q) \) and \( U(x,q) \) both solve (A.1) for all \( x \in (\overline{x}(q), \overline{x}(q)) \), then so does \( F \). Moreover, \( F(x,q) = g(x,q) \) and \( F(\overline{x}(q)) = g(\overline{x}(q),q') \). Thus, by Lemma A1 \( F(x) = E[\epsilon x(q)g(x(q),q')|x_0 = x] \). Moreover, the analysis above implies that,

\[
F'(\overline{x}(q)) > -(1-q')g_x(\overline{x}(q),q') \quad \text{and} \quad F'(\overline{x}(q)) < (\alpha - q)(P_x(\overline{x}(q),q) - 1) + (1-\alpha)V_1'(\overline{x}(q)) = g_x(\overline{x}(q),q').
\]

Suppose by contradiction that \( \overline{x}'(q) \geq \underline{x}'(q) \). Let \( H \) be the solution to (A.1) with \( H(\overline{x}(q)) = g(\overline{x}(q),q') = (1-q')(v_1 - \overline{x}) \) and \( H'(\overline{x}(q)) = g_x(\overline{x}(q),q') = -(1-q') \). By Lemma A3, \( H \) is strictly convex, so \( H'(x) \geq g_x(x,q') = -(1-q') \) and \( H(x) \geq g(x,q') \) for all \( x \in [\overline{x}(q), z_1] \). From S.1 with \( F(x(q),q) = g(x(q),q') \) and \( F'(x(q)) \geq g_x(x(q),q') \), it follows by Lemma A3 that \( F(x) \geq H(x) \geq g(x,q') \) and \( F'(x) \geq H'(x) \geq g_x(x,q') \) for all \( x \in [\overline{x}(q), z_1] \). By Lemma A4, \( L(x,q') \) solves (A.1) on \( (\overline{x}(q), \overline{x}(q)) \), with \( L(x(q),q') = g(x(q),q') \). This, \( F(x(q),q) \) and \( L_x(x(q),q') = g_x(x(q),q') < F'(x(q)) \). Lemma A3 then implies that \( L(x,q') < F(x) = E[\epsilon x(q)g(x(q),q')|x_0 = x] \) for all \( x \in [\overline{x}(q), z_1] \), a contradiction to the fact that \( L(x,q) = \sup x E[\epsilon x(q)g(x(q),q')|x_0 = x] = \overline{x}(q) \). Thus, \( \overline{x}'(q) < \overline{x}(q) \).

Similarly, suppose that \( \overline{x}'(q) \leq \overline{x}(q) \). By a symmetric argument, one can show that \( L(x(q),q') \leq F(x(q),q') \) and \( L_x(x(q),q') = g_x(x(q),q') > F_x(x(q),q') \). Lemma A3 then implies that \( F(x) > L(x,q') \) for all \( x < \overline{x}(q) \), contradicting the fact that \( L(x,q) = \sup x E[\epsilon x(q)g(x(q),q')|x_0 = x] = \overline{x}(q) \). Thus, \( \overline{x}'(q) > \overline{x}(q) \).

Finally, I show that \( L(x,q) \) is strictly convex in \( q \) for all \( x \in (\overline{x}(q), \overline{x}(q)) \). Take \( q' < q < \alpha \), and let \( q_\gamma = \gamma q + (1-\gamma)q' \) for some \( \gamma \in (0,1) \). Note that \( g(x,q_\gamma) = \gamma g(x,q) + (1-\gamma)g(x,q') \). Moreover, \( \overline{x}(q_\gamma) < \overline{x}(q) \) and \( \overline{x}(q') > \overline{x}(q) \). Therefore,

\[
L(x,q_\gamma) = \gamma E \left[ e^{-\epsilon(q_\gamma)}g(x(q_\gamma),q_\gamma) \right]_{x_0 = x} + (1-\gamma) E \left[ e^{-\epsilon(q_\gamma)}g(x(q_\gamma),q_\gamma) \right]_{x_0 = x} < \gamma L(x,q) + (1-\gamma) L(x,q'),
\]

for all \( x \in (\overline{x}(q), \overline{x}(q)) \), so \( L(x,q) \) is strictly convex in \( q \) on \( (\overline{x}(q), \overline{x}(q)) \).
A.3 Proof of Theorem 1

The proof of Theorem 1 is organized as follows. Lemmas A7-A10 establish conditions that hold in any equilibrium. Using these conditions, Lemma A11 establishes that in any equilibrium the monopolist’s profits are equal to $L(x,q)$.

**Lemma A7** Let $(\{q_t\}, P)$ be an equilibrium. Then,

(i) for all $t$ such that $x_t \leq z_2$ and $q_t^- < \alpha$, the monopolist always sells (i.e., $dq_t > 0$),

(ii) for all $t$ such that $x_t > z_2$ and $q_t^- < \alpha$, the monopolist doesn’t sell (i.e., $dq_t = 0$).

**Proof.** (i) Suppose that the monopolist doesn’t sell while $x_t \leq z_2$. Let $\tau = \inf \{s > t : q_s > q_t\}$, so $\tau > t$. By payoff maximization, the price that the marginal buyer $q_t^+$ is willing to pay at time $t$ satisfies $P(x_t, q_t^+) = v_2 - E_t[e^{-r(t-s)}(v_2 - P(x_t, q_t))]$. The monopolist gets a profit margin of $E_t[e^{-r(t-s)}(P(x_\tau, q_\tau) - x_\tau)]$ from selling to $q_t^+$ at time $\tau$, while she would get $P(x_t, q_t^+) - x_t$ from selling to $q_t^+$ at time $t$. Note that

$$P(x_t, q_t^+) - x_t - E_t[e^{-r(t-s)}(P(x_\tau, q_\tau) - x_\tau)] = v_2 - x_t - E_t[e^{-r(t-s)}(v_2 - x_\tau)] > 0,$$

where the inequality follows since $v_2 - x_t > E_t[e^{-r(t-s)}(v_2 - x_\tau)]$ for all $\tau > t$ when $x_t \leq z_2$ (Lemma 1). Thus, the seller is better off selling to $q_t^+$ at $t$. Finally, since $P(x,i)$ has a left-hand limit, there exists $\varepsilon > 0$ such that the monopolist is better off selling to all $i \in (q_t^+, q_t^+ + \varepsilon)$ at time $t$ than after time $\tau$, so $(\{q_t\}, P)$ cannot be an equilibrium.

(ii) Suppose that the monopolist sells while $x_t > z_2$. Let $\tau_\alpha$ denote the time at which consumer $\alpha$ buys and recall that $\tau_\alpha = \inf \{t : x_t \leq z_2\}$. Let $\tau = \min \{\tau_\alpha, \tau_2\}$. I first show that the price at which the monopolist sells at any $s \in [t, \tau_2]$ satisfies $P(x_s, q_s) = v_2 - E_s[e^{-r(t-s)}(v_2 - P(x_\tau, q_\tau))]$. To see this, note that all high types must get the same payoff in equilibrium. Hence, for all $u \in [t, \tau]$ with $u > s$, the price at which the monopolist sells at $s$ satisfies $P(x_s, q_s) = v_2 - E_s[e^{-r(u-s)}(v_2 - P(x_u, q_u))]$. Thus, if $\tau_\alpha \geq \tau_2$, then

$$P(x_s, q_s) = v_2 - E_s[e^{-r(t-s)}(v_2 - P(x_\tau, q_\tau))]$$

for all $s \in [t, \tau]$. On the other hand, if $\tau_\alpha < \tau_2$, then

$$P(x_s, q_s) = v_2 - E_s[e^{-r(t-s)}(v_2 - P(x_\tau, \alpha))].$$

By equation (8),

$$P(x_\tau, \alpha) = v_2 - E_{\tau_\alpha} \left[ e^{-r(t_\tau - \tau_\alpha)}(v_2 - v_1) \right] = v_2 - E_{\tau_\alpha} \left[ e^{-r(t_\tau - \tau_\alpha)}(v_2 - P(x_\tau, \alpha)) \right],$$

where the second equality follows since $P(x_\tau, \alpha) = v_2 - E \left[ e^{-r(t_\tau - \tau_\alpha)}(v_2 - v_1) \right] x_\tau$. Applying the law of iterated expectations and the fact that $q_\tau = \alpha$ whenever $\tau_\alpha < \tau_2$, it follows that $P(x_s, q_s) = v_2 - E_s[e^{-r(t-s)}(v_2 - P(x_\tau, q_\tau))]$.

The profits that the monopolist gets from selling to high valuation consumers between time $t$ and $\tau_2$ are $E_t[e^{-r(s-t)} \int_t^{\tau_2} (P(x_s, q_s) - x_s)dq_s]$. If instead the monopolist waits until time $\tau_2$ and sells to all consumers $i \in [q_t^-, q_\tau]$ at that instant, her profits are $E_t[e^{-r(t_\tau - t)}(P(x_\tau, q_\tau) - x_\tau)]$. Since $P(x_s, q_s) = v_2 - E_s[e^{-r(t_\tau - s)}(v_2 - P(x_\tau, q_\tau))]$ for all $s \in [t, \tau_2]$,

$$P(x_s, q_s) - x_s - E_s \left[ e^{-r(t_\tau - s)}(P(x_\tau, q_\tau) - x_\tau) \right] = v_2 - x_s - E_s \left[ e^{-r(t_\tau - s)}(v_2 - x_\tau) \right] < 0.$$
since, by Lemma 1, \(v_2 - x_s < E_s[e^{-r(t_2-t_1)}(v_2 - x_{t_2})]\) for all \(x_s > z_2\). Hence, the seller is better off by delaying sales until \(t_2\), so \((\{q_t\}, P)\) cannot be an equilibrium. ■

**Lemma A8** Let \((\{q_t\}, P)\) be an equilibrium and let \(\Pi(x, q)\) be the seller’s profits. If \(\{q_s\}\) is continuous in \([t, \tau]\) for some \(\tau > t\), then \(\Pi(x_t, q_t) = E_t[e^{-r(u-t)}\Pi(x_u, q_t)]\) for all \(u \in [t, \tau]\).

**Proof.** Suppose first that \(\{q_s\}\) is constant on \([t, \tau]\). Then, \(\Pi(x_t, q_t) = E_t[e^{-r(u-t)}\Pi(x_u, q_t)]\) for all \(u \in [t, \tau]\), since the monopolist doesn’t make any sales on \([t, \tau]\).

Suppose next that \(\{q_s\}\) is continuous and strictly increasing in \([t, \tau]\). Fix \(u \in [t, \tau]\). Since the monopolist can always choose to make no sales between \(t\) and \(u > t\), \(\Pi(x_t, q_t) \geq E_t[e^{-r(u-t)}\Pi(x_u, q_t)]\). On the other hand, \(\Pi(x, q) \geq \int_q^u (P(x, i) - x)di + \Pi(x, q')\) for all \((x, q)\) and all \(q' > q\), since the monopolist can get profits arbitrarily close to \(\int_q^u (P(x, i) - x)di + \Pi(x, q')\) by selling to all buyers \(i \in [q, q']\) arbitrarily fast. Suppose by contradiction that \(\Pi(x_t, q_t) > E_t[e^{-r(u-t)}\Pi(x_u, q_t)]\) for some \(u \in [t, \tau]\). Therefore, it must be that

\[
\Pi(x_t, q_t) = E_t \left[ \int_t^u e^{-r(s-t)} (P(x_s, q_s) - x_s) dq_s + e^{-r(u-t)}\Pi(x_u, q_u) \right] > E_t \left[ e^{-r(u-t)}\Pi(x_u, q_t) \right].
\]

Since \(\Pi(x_u, q_t) \geq \int_q^u (P(x_u, i) - x_u)di + \Pi(x_u, q_u)\), it follows that

\[
E_t \left[ \int_t^u e^{-r(s-t)} (P(x_s, q_s) - x_s) dq_s \right] > E_t \left[ e^{-r(u-t)} \int_q^u (P(x_u, i) - x_u) di \right]. \tag{A.4}
\]

Equation (A.4) in turn implies that there exists a set of positive measure \([q, \overline{q}] \subset [q_t, q_s]\) and \(s_1, s_2 \in [t, u]\) with \(s_2 > s_1\) such that \(P(x_s, i) - x_s > E_s[e^{-r(s'-s)}(P(x_{s'}, i) - x_{s'})]\) for all \(i \in [q, \overline{q}]\) and all \(s, s' \in [s_1, s_2]\). Pick \(\varepsilon > 0\) small enough such that \(\tau_\varepsilon = \inf\{s : q_s \geq q + \varepsilon\} \leq s_2\) whenever the state at time \(s_1 = (x_s, q)\). Then, it follows that

\[
\Pi(x_{s_1}, q) = E_{s_1} \left[ \int_{s_1}^{s_2} e^{-r(s-s_1)} (P(x_s, q_s) - x_s) dq_s + e^{-r(s_2-s_1)}\Pi(x_{s_2}, q_{s_2}) \right] < \int_{s_1}^{s_2+\varepsilon} (P(x_s, i) - x_s) di + E_{s_1} \left[ e^{-r(s_2-s_1)}\Pi(x_{s_2}, q_{s_2}) \right]. \tag{A.5}
\]

At state \((x_{s_1}, q)\) the monopolist can get a payoff arbitrarily close to the right-hand side of (A.5) by selling to all \(i \in [q, q + \varepsilon]\) arbitrarily fast and then not making any sales until time \(\tau_\varepsilon\). Thus, the seller has a strategy that yields her a higher payoff than \(\{q_t\}\), which cannot be since \((\{q_t\}, P)\) is an equilibrium. Hence, \(\Pi(x_t, q_t) = E_t[e^{-r(s-t)}\Pi(x_s, q_s)]\) for all \(s \in [t, \tau]\). ■

**Lemma A9** Let \((\{q_t\}, P)\) be an equilibrium and let \(\Pi(x, q)\) be the seller’s profits. Let \((x, q)\) with \(q < \alpha\) be such that \(\{q_t\}\) is continuous at time \(s\) when \((x_s, q_s) = (x, q)\). Then, there exists \(\tau > s\) such that \(\{q_t\}\) jumps at state \((x, q, \tau)\).

Moreover,

\[
\Pi(x_s, q_s) = E_s[e^{-r(s-t)} \left( (P(x_{t}, q + dq_{t}) - x_{t}) dq_{t} + \Pi(x_{t}, q + dq_{t}) \right)], \tag{A.6}
\]
where \( dq_t \) denotes the jump of \( \{q_t\} \) at state \((x_\tau, q)\).

**Proof.** Let \((x_s, q_s) = (x, q)\) be as in the statement of the Lemma. By Lemma A8, there exists \( \tau > s \) such that \( \Pi(x_s, q_s) = E_s[e^{-r(\tau-s)}\Pi(x_\tau, q_s)] \). There are two possibilities: (i) \( \{q_t\} \) jumps at state \((x_\tau, q_s)\), or (ii) \( \{q_t\} \) is continuous at such state. In case (i), it must be that \( \Pi(x_\tau, q_s) = (P(x_\tau, q + dq_\tau) - x_\tau) dq_\tau + \Pi(x_\tau, q + dq_\tau) \), where \( dq_\tau \) denotes the jump of \( \{q_t\} \) at state \((x_\tau, q_s)\). Thus, \( \Pi(x_s, q_s) = E_s[e^{-r(\tau-s)}((P(x_\tau, q + dq_\tau) - x_\tau) dq_\tau + \Pi(x_\tau, q + dq_\tau))], \) and so the statement of the Lemma is true.

Consider next case (ii), so that \( \{q_t\} \) is continuous at state \((x_\tau, q_s)\). By Lemma A8 there exists \( \tau' > \tau \) such that \( \Pi(x_\tau, q_s) = E_e[e^{-r(\tau'-\tau)}\Pi(x_{\tau'}, q_s)] \), so by the law of iterated expectations \( \Pi(x_s, q_s) = E_s[e^{-r(\tau'-s)}\Pi(x_{\tau'}, q_s)] \). Again, there are two possibilities: \( \{q_t\} \) jumps at state \((x_{\tau'}, q_s)\), or (ii) \( \{q_t\} \) is continuous at such state. By the argument above, the statement of the Lemma is true if the relevant case is (i). Otherwise, in case (ii) there exists \( \tau'' > \tau' \) such that \( \Pi(x_s, q_s) = E[e^{-r(\tau''-s)}\Pi(x_{\tau''}, q_s)] \). Continuing this way, there must exist \( \bar{\tau} \) such that \( \{q_t\} \) jumps at state \((x_{\bar{\tau}}, q_s); \) otherwise, \( \Pi(x_s, q_s) = \lim_{\tau \to \infty} E[e^{-r(\tau-s)}\Pi(x_\tau, q_s)] \big| x_s = 0, \) which contradicts \( \Pi(x_s, q_s) \geq L(x_s, q_s) > 0. \)

**Lemma A10** Let \( \{q_t\}, P \) be an equilibrium and let \( \Pi(x, q) \) be the seller’s profits. If \( \{q_s\} \) is continuous and strictly increasing in \([t, \tau]\) for some \( \tau > t \), then \( -\Pi_y(x_s, q_s) = P(x_s, q_s) - x_s \) for all \( s \in [t, \tau) \).

**Proof.** Note first that for all \( \varepsilon > 0 \) and for all \( s \in [t, \tau) \), it must be that

\[
\Pi(x_s, q_s) \geq (P(x_s, q_s + \varepsilon) - x_s) \varepsilon + \Pi(x_s, q_s + \varepsilon), \tag{A.7}
\]

since the monopolist can always choose at time \( s \) to sell to all buyers \( i \in [q_s, q_s + \varepsilon] \) at price \( P(x_s, q_s + \varepsilon) \). Next, I show that for all \( s \in [t, \tau) \) it must also be that

\[
\Pi(x_s, q_s) \leq (P(x_s, q_s) - x_s) \varepsilon + \Pi(x_s, q_s + \varepsilon), \tag{A.8}
\]

for all \( \varepsilon > 0 \) small enough. To see this, let \( \tau_\varepsilon = \inf\{u : q_u \geq q_s + \varepsilon\} \), so that

\[
\Pi(x_s, q_s) = E_s \left[ \int_s^{\tau_\varepsilon} e^{-r(u-s)} (P(x_u, q_u) - x_u) dq_u + e^{-r(\tau_\varepsilon-s)} \Pi(x_{\tau_\varepsilon}, q_s + \varepsilon) \right].
\]

Since \( \Pi(x_s, q_s + \varepsilon) \geq E_s [e^{-r(\tau_\varepsilon-s)} \Pi(x_{\tau_\varepsilon}, q_s + \varepsilon)] \) and since \( P(x_s, q_s) \geq P(x_s, i) \) for all \( i > q_s \),

\[
\Pi(x_s, q_s) - (P(x_s, q_s) - x_s) \varepsilon - \Pi(x_s, q_s + \varepsilon) \\
\leq E_s \left[ \int_s^{\tau_\varepsilon} e^{-r(u-s)} (P(x_u, q_u) - x_u) dq_u \right] - \int_{q_s + \varepsilon}^{q_s} (P(x_s, i) - x_s) di, \tag{A.9}
\]

To establish (A.8) it suffices to show that the right-hand side of (A.9) is less than zero. Towards a contradiction, suppose that \( E_s[\int_s^{\tau_\varepsilon} e^{-r(u-s)}(P(x_u, q_u) - x_u) dq_u] > \int_{q_s + \varepsilon}^{q_s + \varepsilon} (P(x_s, i) - x_s) di \)
other hand, if \(x_s\)di. Then, there exists \(s_1, s_2 \in [s, \tau_x]\) with \(s_1 < s_2\) and a set \([q, \bar{q}] \subset [q_x, q_\varepsilon + \varepsilon]\) such that

\[
P(x_{s_1}, i) - x_{s_1} < E_{s_1} \left[ e^{-r(s' - s_1)} (P(x_{s'}, i) - x_{s'}) \right],
\]

(A.10)

for all \(i \in [q, \bar{q}]\) and all \(s' \in (s_1, s_2]\). Suppose that the state at time \(s_1\) is \((x_{s_1}, q)\), and fix \(\tau' \in (s_1, s_2]\) such that \(q_{\tau'} \leq \bar{q}\). Equation (A.10) implies that

\[
\Pi(x_{s_1}, q) = E_{s_1} \left[ \int_{s_1}^{\tau'} e^{-r(s - s_1)} (P(x_s, q_s) - x_s) dq_s + e^{-r(\tau' - s_1)} \Pi(x_{\tau'}, q_{\tau'}) \right]
\]

\[
< E_{s_1} \left[ e^{-r(\tau' - s_1)} \left( \int_{q}^{q_{\tau'}} (P(x_{\tau'}, i) - x_{\tau'}) di + \Pi(x_{\tau'}, q_{\tau'}) \right) \right].
\]

(A.11)

Note that at state \((x_{s_1}, q)\) the seller can get a payoff arbitrarily close to the right-hand side of (A.11) by not making any sales until time \(\tau'\), and then selling to all buyers \(i \in [q, q_{\tau'}]\) arbitrarily fast at time \(\tau'\). Thus, at state \((x_{s_1}, q)\) the seller has a strategy different from \(\{q_x\}\) that gives her a larger payoff than \(\Pi(x_{s_1}, q)\), contradicting the assumption that \((\{q_x\}, \mathcal{P})\) is an equilibrium. Hence, (A.8) must hold for \(\varepsilon\) small enough.

Finally, by (A.7) and (A.8), for all \(\varepsilon > 0\) small enough,

\[
P(x_s, q_x + \varepsilon) - x_s \leq -\frac{\Pi(x_s, q_x + \varepsilon) - \Pi(x_s, q_x)}{\varepsilon} \leq P(x_s, q_x) - x_s.
\]

Since \(P(x, q)\) must be continuous in \(q\) at \((x_s, q_x)\) (otherwise, prices would jump down at time \(s\), and those consumers who buy at \(s^-\) would be strictly better off by waiting), it follows that

\[-\Pi_q(x_s, q_x) = P(x_s, q_x) - x_s.\]

**Lemma A11** Let \((\{q_x\}, \mathcal{P})\) be an equilibrium, and let \(\Pi(x, q)\) denote the monopolist’s profits. Then, \(\Pi(x, q) = L(x, q)\) for all states \((x, q)\) with \(q < \alpha\).

**Proof.** By the arguments in the main text, \(\Pi(x, q) \geq L(x, q)\) for all states \((x, q)\) with \(q < \alpha\). By Lemma A9, for all \((x, q)\) such that \(\{q_x\}\) is continuously increasing at time \(s\) when \((x_s, q_s) = (x, q)\), there exists \(\tau > s\) and \(dq_\tau > 0\) such that

\[
\Pi(x, q) = E_s \left[ e^{-r(\tau - s)} ((P(x_\tau, q + dq_\tau) - x_\tau) dq_\tau + \Pi(x_\tau, q + dq_\tau)) \right].
\]

If \(q + dq_\tau = \alpha\), then \(\Pi(x, q) = E_s \left[ e^{-r(\tau - s)} g(x_\tau, q) \right] \leq L(x, q)\), so \(\Pi(x, q) = L(x, q)\). On the other hand, if \(dq_\tau + q = \bar{q} < \alpha\), then \(\Pi(x_\tau, q) = (P(x_\tau, \bar{q}) - x_\tau)(\bar{q} - q) + \Pi(x_\tau, \bar{q})\). By Lemma A7, \(\{q_t\}\) must be continuous and strictly increasing after time \(\tau\) (since, by Lemma A7, \(x_\tau \leq z_2\)). Lemma A10 then implies that \(P(x_\tau, \bar{q}) - x_\tau = -\Pi_q(x_\tau, \bar{q})\). By Lemma A9, \(\Pi(x_\tau, q) = E_\tau \left[ e^{-r(\tau' - \tau)} ((P(x_\tau, q + dq_\tau) - x_\tau) dq_\tau + \Pi(x_\tau, q + dq_\tau)) \right] \) for some \(\tau' > \tau\), where \(dq_\tau\) denotes the jump of \(\{q_t\}\) at state \((x_\tau, \bar{q})\). Therefore, \(P(x_\tau, q) - x_\tau = -\Pi_q(x_\tau, \bar{q}) = E_\tau \left[ e^{-r(\tau' - \tau)} ((P(x_\tau, q + dq_\tau') - x_\tau) dq_\tau') \right] \) for some \(\tau' > \tau\), where \(dq_\tau\) denotes the jump of \(\{q_t\}\) at state \((x_\tau, \bar{q})\). Therefore, \(\Pi(x_\tau, q) = (P(x_\tau, \bar{q}) - x_\tau)(\bar{q} - q) + \Pi(x_\tau, \bar{q})\), it
follows that
\[ \Pi(x, q) = E_x \left[ e^{-r(t'-\tau)} \left( (P(x, q) - x_{t'}) (dq_{t'} + \tilde{q} - q) + \Pi(x_{t'}, q_{t'}) \right) \right], \quad (A.12) \]

where \( q_{t'} = \tilde{q} + dq_{t'} \). There are two possibilities: (i) \( q_{t'} = \alpha \), or (ii) \( q_{t'} < \alpha \). In the first case, \( P(x_{t'}, q_{t'}) = g(x_{t'}, q) \). Using (A.12), this implies that
\[ \Pi(x, q) = E_x \left[ e^{-r(t'-\tau)} g \left( x_{t'}, q \right) \right]; \]
and since \( \Pi(x, q) = E_s[ e^{-r(\tau-s)} \Pi(x, q) ] \), by the Law of Iterated Expectations, \( \Pi(x, q) = E_s[ e^{-r(\tau-s)} g \left( x_{t'}, q \right) ] \leq L(x, q) \), so \( \Pi(x, q) = L(x, q) \). In the second case, \( q_{t'} < \alpha \). Since \( x_{t'} = z_2 \), by Lemma A7 the monopolist must continue selling gradually at state \( (x_{t'}, q_{t'}) \). Then, by Lemma A9, there exists \( \tau'' \) such that
\[ \Pi(x_{t'}, q_{t'}) = E_{x_{t'}} \left[ e^{-r(\tau''-\tau')} \left( (P(x_{t'}, q_{t''}) - x_{t''}) (dq_{t''} - q_{t'}) + \Pi(x_{t''}, q_{t''}) \right) \right], \quad (A.13) \]

where \( q_{t''} = dq_{t'} + q_{t'} \). Moreover, the same arguments as above imply that \( P(x_{t'}, q_{t'}) = E_{x_{t'}}[ e^{-r(\tau''-\tau')}(P(x_{t''}, q_{t''}) - x_{t''})] \). Using this and (A.13) in (A.12), it follows that
\[ \Pi(x, q) = E_x \left[ e^{-r(\tau'-\tau)} \left( (P(x, dq_{t'} + q) - x_{t'}) dq_{t'} + \Pi(x_{t'}, dq_{t'} + q) \right) \right], \]
for some stopping time \( \tau' \) and some \( dq_{t'} > 0 \). Again, there are two possibilities: (i) \( dq_{t'} + q = \alpha \), or (ii) \( dq_{t'} + q < \alpha \). In case (i), \( (P(x_{t'}, dq_{t'} + q) - x_{t'}) dq_{t'} + \Pi(x_{t'}, dq_{t'} + q) = g(x_{t'}, q) \), so \( \Pi(x, q) = E_x \left[ e^{-r(\tau'-\tau)} g(x_{t'}, q) \right] \). By the Law of Iterated Expectations, \( \Pi(x, q) = E_s[ e^{-r(\tau-s)} g(x_{t'}, q) ] \leq L(x, q) \), so \( \Pi(x, q) = L(x, q) \). In case (ii), we can again repeat the same argument. Eventually, there will be \( \tau'' \) such that \( \Pi(x, q) = E_s[ e^{-r(\tau-s)} g(x_{t'}, q) ] \leq L(x, q) \) for some stopping time \( \tau' \), so \( \Pi(x, q) = L(x, q) \). \( \blacksquare \)

**Proof of Theorem 1.** Lemma A11 shows that, in any equilibrium, the monopolist’s profits are \( L(x, q) \) for all \( (x, q) \). In order for the monopolist to obtain profits equal to \( L(x, q) \), she must sell to all high types when \( x_t \in [0, \bar{x}(q_i)] \cup [\bar{x}(q_{-i}), z_2] \) (and also to low types when \( x_t \leq x(q_{-i}) \)). Moreover, by Lemma A7 the monopolist must sells at a positive rate while \( x_t \in (x(q_i), \bar{x}(q_i)) \). The arguments in the text pin down the rate at which the monopolist sells and the price she charges when \( x_t \in (x(q_i), \bar{x}(q_i)) \). \( \blacksquare \)

**A.4 Proof of Proposition 3**

Fix a sequence \( (\sigma_n, \mu_n) \to 0 \). For each \( n \), let \( \lambda_n \) be the negative root of \( \frac{1}{2} \sigma_n^2 g (y - 1) + \mu_n y = r \), and for \( i = 1, 2 \) let \( z_i^n = \frac{-\lambda_n}{1-\lambda_n} \). One can show that \( \lim_{n \to \infty} \lambda_n = -\infty \), so \( \lim_{n \to \infty} z_i^n = v_i \). Let \( P^n(x, \alpha) \) be the price consumer \( \alpha \) is willing to pay when \( (\sigma_n, \mu_n) \). By equation (9), it follows that \( \lim_{n \to \infty} P^n(x, \alpha) = v_1 \) for all \( x \leq v_1 \) and \( \lim_{n \to \infty} P^n(x, \alpha) = v_2 \) for all \( x > v_1 \).

By Theorem 1, for each \( n \) there exists \( x^n(0) < z_i^n \) and \( \bar{x}^n(0) \in (z_i^n, z_i^\alpha) \) such that the monopolist sells at \( t = 0 \) to all high types at price \( P^n(x, \alpha) \) whenever \( x_n \in [\bar{x}^n(0), z_i^n] \), and sells at \( t = 0 \) to all buyers (high and low types) at price \( v_1 \) whenever \( x_n \leq z_i^n \). Thus, to prove Proposition 3 it suffices to show that \( \bar{x}^* = \lim_{n \to \infty} \bar{x}^n(0) = v_1 \) and \( \bar{x}^* = \lim_{n \to \infty} \bar{x}^n(0) = v_1 \) (if either of these limits don’t exist, take a convergent subsequence).
Since \( \lim_{n \to \infty} z^n_i = v_i \) for \( i = 1, 2, \) and since \( \tilde{x}^n(0) < z^n_i \) and \( \tilde{x}^n(0) \in (z^n_i, z^n_j) \) for all \( n, \) it follows that \( \tilde{x}^* \leq v_1 \) and \( \tilde{x}^* \geq v_1. \) Suppose by contradiction that \( \tilde{x}^* \neq v_1 \) and/or \( \tilde{x}^* \neq v_1, \) so that \( \tilde{x}^* < \tilde{x}^n. \) Thus, there exists \( N \) and \( y < \tilde{y} \) such that \( x^n(0) \leq y \) and \( \tilde{x}^n(0) \geq \tilde{y} \) for all \( n \geq N. \) Let \( L^n(x, 0) \) be the monopolist’s profits at state \( (x, 0) \) when \( (\sigma, \mu) = (\sigma_n, \mu_n). \) By Theorem 1, \( L^n(x, 0) = E^n[e^{-r\tau^n(0)}(P^n(x_{r_n}(0), k_n) - x_{r_n}(0)) + \Pi^n(x_{r_n}(0), k_n))] \), where \( \tau^n(0) = \inf\{t : x_t \in [0, \tilde{x}^n(0)] \cup [\tilde{x}^n(0), z^n]\} \) and where \( E^n[\cdot] \) and \( \Pi^n(x, k) \) denote, respectively, the expectation operator and the seller’s profits at state \( (x, k) \) when \( (\sigma, \mu) = (\sigma_n, \mu_n). \) Let \( \tilde{\gamma} = \inf\{t : x_t \notin (y, \tilde{y})\} \) and note that for all \( n \geq N, \tau^n(0) \geq \tilde{\gamma} \) whenever \( x_0 \in [y, \tilde{y}]. \) Fix \( x \in [y, \tilde{y}]. \) Since \( P^n(x_{r_n}(0), k_n) < v_2 \) and \( \Pi^n(x_{r_n}(0), k_n) < (1 - \alpha) \) \( v_2 \) for all \( n, \) it follows that \( L^n(x, 0) < v_2 E^n[e^{-r\tilde{\gamma}}] \) for all \( n \geq N. \) Finally, note that \( E^n[e^{-r\tilde{\gamma}}] \to 0 \) as \( n \to \infty \) whenever \( x_0 \in (y, \tilde{y}), \) (as \( (\sigma, \mu) \to 0, \) it takes arbitrarily long until costs leave the interval \((y, \tilde{y})\)). This implies that \( L^n(x, 0) \to 0, \) which cannot be since \( L^n(x, 0) \geq \alpha P^n(x, k) - x \) \( + (1 - \alpha) \Pi^n(x, k) \) \( > 0 \) for all \( x < z_2^n \) and all \( n. \) Thus, \( \tilde{x}^* = \tilde{x}^n = v_1. \)

A.5 Proof of Theorem 2

The proof of Theorem 2 generalizes the proof of Theorem 1. Here I provide a sketch of the arguments. Suppose that there are \( n \geq 3 \) types of buyers in the market, with valuations \( v_1 < \ldots < v_n. \) For \( k = 1, \ldots, n, \) let \( a_k = \max\{i \in [0, 1] : f(i) = v_k\} \) be the highest indexed consumer with valuation \( v_k. \) Note that for \( q \geq a_3, \) the only buyers left in the market are those with valuations \( v_1 \) and \( v_2. \) Then, by Theorem 1 the monopolist’s equilibrium profits are \( L(x, q) = \sup_{x \in T} E[e^{-r\tau}(P(x, \alpha_3) - x)(\alpha_2 - q) + (1 - \alpha_2) V_1(x, q) | x_0 = x] \) for all states \((x, q)\) with \( q \geq a_3. \)

Consider next states \((x, q)\) with \( q \in (a_4, a_3), \) so there are \( a_3 - q \) buyers with valuation \( v_3 \) in the market (if there are only three types of buyers in the market, let \( a_4 = 0). \) By equation (5), the strategy \( P(x, a_3) \) (the highest indexed consumer with valuation \( v_3) \) satisfies \( P(x, a_3) = v_3 - E[e^{-r\tau_2}(v_3 - P(x_{r_2}, q_{r_2})) | x], \) where \( \tau_2 = \inf\{t : x_t \leq z_2\} \) is the time at which the monopolist starts selling to buyers with valuation \( v_2 \) when the level of market penetration is \( a_3. \) By the skimming property, the monopolist can always sell to all buyers with valuation \( v_3 \) at price \( P(x, a_3). \) Therefore, at states \((x, q)\) with \( q \in (a_4, a_3) \) the monopolist’s profits are bounded below by

\[
L(x, q) = \sup_{x \in T} E[e^{-r\tau}(P(x, a_3) - x)(\alpha_3 - q) + e^{-r\tau} L(x, a_3) | x_0 = x]. \tag{A.14}
\]

Using arguments similar to those in Lemma A4, one can show that the solution to (A.14) is of the form \( \tau(q) = \inf\{t : x_t \in [0, \underline{x}_1(q)] \cup [\underline{x}_1(q), \underline{x}_2(q)] \cup [\underline{x}_2(q), \underline{x}_3(q)]\}, \) with \( \underline{x}_1(q), \underline{x}_1(q), \underline{x}_2(q), \) \( \underline{x}_2(q) \) such that \( \underline{x}_1(q) < z_1 < \underline{x}_1(q) \) \( < \underline{x}_2(q) < z_2 < \underline{x}_2(q) < z_3. \) That is, the solution to (A.14) involves delaying when \( x \) is around \( z_1 \) or \( z_2 \) and when \( x > z_3. \) Moreover, by arguments similar to those in Lemma A5, \( L(x, q) \in C^{2,2} \) for all \( x \in (\underline{x}_1(q), \underline{x}_1(q)) \cup (\underline{x}_2(q), \underline{x}_2(q)). \)

Next, by arguments similar to those in Lemma A7 the monopolist always sells at a positive rate (i.e., \( dq_t > 0 \)) at states \((x_t, q_t)\) with \( x_t \leq z_3 \) and \( q_t \in [a_4, a_3), \) and never sells at states with \( x_t > z_3 \) and \( q_t \in [a_4, a_3). \) Moreover, arguments similar to those in Lemma A10 imply that in any equilibrium, \( P(x, q) = x - \Pi_q(x, q) \) whenever the monopolist is selling at
a continuous rate (i.e., whenever \( q_t \) is continuous and strictly increasing). Finally, by arguments similar to those in Lemma A11, in any equilibrium the monopolist’s profits must be equal to \( L(x, q) \) at all states \((x, q)\) with \( q \in [\alpha_4, \alpha_3] \).

At states \((x_t, q_{t-})\) with \( q_{t-} \in [\alpha_4, \alpha_3] \) the equilibrium dynamics are as follows. If \( x_t > z_3 \), the monopolist doesn’t sell and waits for costs to decrease. When \( x_t \in [0, x_1(q_{t-})] \cup [x_1(q_{t-}), x_2(q_{t-})] \cup [x_2(q_{t-}), z_3] \) (i.e., when \( x_t \) is in the stopping region of (A.14)), the monopolist sells immediately to all remaining consumers with valuation \( v_3 \) at price \( P(x_t, \alpha_3) \), and then equilibrium play continues as in the case with two types of consumers. Finally, when \( x_t \in (x_1(q_{t-}), x_2(q_{t-})) \cup (x_2(q_{t-}), x_3(q_{t-})) \), the monopolist sells gradually to consumers with valuation \( v_3 \) at price \( P(x_t, q_t) = x_t - L_q(x_t, q_t) \). In this region, prices evolve in such a way that buying with valuation \( v_3 \) are indifferent between trading at \( t \) or waiting.

Consider next states \((x, q)\) with \( q \in [\alpha_4, \alpha_4] \), at which there are \( \alpha_4 - q \) buyers with valuation \( v_4 \) in the market (if there are only four types of buyers, let \( \alpha_5 = 0 \)). Let \( P(x, \alpha_4) \) be the strategy of consumer \( \alpha_4 \) (i.e., the highest indexed consumer with valuation \( v_4 \)). Equation (5) implies that \( P(x, \alpha_4) = v_4 - E[e^{-r\tau_3} (v_4 - P(x_{\tau_3}, q_{\tau_3}))|x] \), where \( \tau_3 = \inf\{t : x_t \leq z_3\} \) is the time at which the monopolist sells to buyers with valuation \( v_3 \) when \( q = \alpha_4 \). Since the monopolist can sell to all buyers with valuation \( v_4 \) at price \( P(x, \alpha_4) \), at states \((x, q)\) with \( q \in [\alpha_5, \alpha_4] \) her profits are bounded below by \( L(x, q) = \sup_{\tau \in \mathcal{T}} E[e^{-r\tau} g(x_{\tau}, q)|x] \), where \( g(x, q) = (P(x, \alpha_4) - x_{\tau}) (\alpha_4 - q) + L(x, \alpha_4) \). Repeating the same arguments, in any equilibrium the monopolist’s profits are \( L(x, q) \) at all states \((x, q)\) with \( q \in [\alpha_5, \alpha_4] \). More generally, for \( k \geq 5 \) one can extend \( L(x, q) \) for all \( q \in [\alpha_k+1, \alpha_k] \) in a similar way, and show that in equilibrium the seller’s profits are \( L(x, q) \) for all \( q \in [\alpha_k+1, \alpha_k] \).

### A.6 Proof of Theorem 3

Fix a sequence \( \{f^j\} \to h \). For \( j = 1, 2, \ldots \), let \( v_1^j < v_2^j < \ldots < v_n^j \) be the (finite) set of possible valuations under \( f^j \). For \( k = 1, \ldots, n_j \), let \( z_k^j = \frac{1}{\beta^j} v_k^j \). For each \( j \), define the function \( P^j(x) \) as follows. For \( x \leq z_1^j \), \( P^j(x) = v_1^j \). For \( k = 2, 3, \ldots, n_j \) and \( x \in (z_k^{j-1}, z_k^j) \), \( P^j(x) = P^j(x, \alpha_k^j) \). That is, for all \( x \in (z_k^{j-1}, z_k^j) \), \( P^j(x) \) is equal to the price consumer \( \alpha_k^j \) is willing to pay (where \( \alpha_k^j = \max\{i \in [0, 1] : f^j(i) = v_i^j\} \)). By equation (5), for \( k = 2, \ldots, n_j \) and \( x \in (z_{k-1}^j, z_k^j) \),

\[
P^j(x) = P^j(x, \alpha_k^j) = v_k^j - E\left[e^{-\tau_{k-1}} (v_k^j - P^j(z_{k-1}^j, \alpha_k^j))\right]_{x_0 = x}
\]

\[
= v_k^j - E\left[e^{-\tau_{k-1}} (v_k^j - P^j(z_{k-1}^j))\right]_{x_0 = x},
\]

(A.15)

where for \( k = 1, \ldots, n_j \), \( \tau_k^j = \inf\{t : x_t \leq z_k^j\} \) is the time at which the monopolist starts selling to buyers with valuation \( v_k^j \) when \( v_k^j \) is the highest valuation remaining in the market.

**Lemma A12** For \( k = 2, 3, \ldots \) and \( x \in (z_{k-1}^j, z_k^j) \), \( P^j(x) = v_k^j - \sum_{m=1}^{k-1} (v_{m+1}^j - v_m^j) (x/\beta^j)^m \).

**Proof.** The proof is by induction. By equation (9), \( P^j(x) = v_1^j - (v_2^j - v_1^j) (x/\beta^j)^1 \) for \( x \in (z_1^j, z_2^j) \), so the statement is true for \( k = 2 \). Suppose the statement is true for \( l = 2, \ldots, k-1 \).
By Corollary A1, $P^j (x) = v^j_2 - (v^j_2 - P^j (z^j_{k-1})) (x/z^j_{k-1})^\lambda$ for all $x \in (z^j_{k-1}, z^j_k]$. The induction hypothesis then implies that

$$P^j (x) = v^j_k - (v^j_k - P^j (z^j_{k-1})) (x/z^j_{k-1})^\lambda = v^j_k - \frac{1}{\lambda} \left( \frac{v^j_k}{v^j_k} \right)^\lambda,$$

for $x \in (z^j_{k-1}, z^j_k]$. \[\square\]

Let $z = \frac{1}{1-\lambda} \tau$ and $\tilde{z} = \frac{1}{1-\lambda} v$, and let $V(x) = \sup_{\tau} E[e^{-\tau \rho} (v - x)] | x_0 = x]$. By Lemma 1, $V(x) = E[e^{-\tau \rho} (v - x) | x_0 = x]$, where $\tau = \inf \{ t : x_t \leq z \}$.

**Lemma A13** $P^j (x) - x \to V(x)$ uniformly on $[0, \tilde{z}]$ as $j \to \infty$.

**Proof.** I first show that $\lim_{j \to \infty} P^j (x) = V(x) + x$ for all $x \in [0, \tilde{z}]$. Note first that, for all $x \leq \tilde{z}$, $\lim_{j \to \infty} P^j (x) = \lim_{j \to \infty} v^j_1 = v = V(x) + x$. Next, fix $x \in (z, \tilde{z})$ and for $j = 1, 2, ..., \text{let } k_j \text{ be such that } x \in (z^j_{k_j-1}, z^j_{k_j})$. Let $v^j = \frac{1}{1-\lambda} x$. Equation (A.16) and the fact that $x/z^j_m = v(x)/v^j_m$ imply that $P^j (x) = v^j_k - \frac{1}{\lambda} \left( v^j_k \right)^\lambda$. Since $x \in (z^j_{k_j-1}, z^j_{k_j})$ for all $j$ and since $\lim_{j \to \infty} z^j_{k_j-1} - z^j_{k_j} = 0$, it follows that $z^j_{k_j} = \frac{1}{1-\lambda} v^j_k \to x$ as $j \to \infty$. Hence, $\lim_{j \to \infty} v^j_{k_j} = \frac{1}{1-\lambda} x = v(x)$. Since $(v(x)/v)^\lambda$ is Riemann integrable in $v$,

$$\lim_{j \to \infty} P^j (x) = v(x) - \int_v^x (v(x)/v)^\lambda dv = x + (v - \tilde{z}) \frac{(v/\tilde{z})^\lambda}{\lambda} = x + V(x).$$

Finally, since $P^j (x)$ is increasing in $x$ for all $j$ and since $\lim_{j \to \infty} P^j (x) = V(x) + x$ for all $x \in [0, \tilde{z}]$, it follows that $P^j (x) \to V(x) + x$ uniformly on $[0, \tilde{z}]$ as $j \to \infty$. Thus, $P^j (x) - x \to V(x)$ uniformly on $[0, \tilde{z}]$ as $j \to \infty$. \[\square\]

**Proof of Theorem 3.** I first prove that, for all $x$, $L^j (x, 0) \to V(x)$ as $j \to \infty$. Note first that $L^j (x, 0) \geq V(x)$ for all $x$, since at any state $(x, 0)$ the monopolist can wait until time $\tau^j_1$ and sell to all buyers at price $v^j_1 \geq v$, obtaining a profit of $E[e^{-\tau^j_1 (v^j_1 - x_{\tau^j_1})} | x_0 = x] \geq V(x)$.

Next, consider the case in which $x_0 = x \geq \tilde{z}$. In this case, in equilibrium the monopolist sells to consumers with valuation $v^j_k$ at time $\tau^j_k = \inf \{ t : x_t \leq z^j_k \}$ (for $k = 1, ..., n_j$) at a price $P(z^j_k, \alpha^j_k) = P^j (z^j_k)$ (where $\alpha^j_k = \max \{ i : f^j (i) = v^j_k \}$). Let $\alpha^j_{k_j+1} = 0$. Then, the seller’s profits are $L^j (x, 0) = E[\sum_{k=1}^{n_j} e^{-\tau^j_k} P^j (z_k - z^j_k) (\alpha^j_k - \alpha^j_{k+1})]$. Since $P^j (x) - x \to V(x)$ uniformly on $[0, \tilde{z}]$ as $j \to \infty$, for every $\eta > 0$ there exists $N$ such that $P^j (x) - x - V(x) < \eta$ for all $j > N$ and all $x \in [0, \tilde{z}]$. Thus, for $j > N$,

$$L^j (x, 0) < E \left[ \sum_{k=1}^{n_j} e^{-\tau^j_k} V(x_{\tau^j_k}) | x_0 = x \right] + \eta = \sum_{k=1}^{n_j} \alpha^j_k E \left[ e^{-\tau^j_k} V(x_{\tau^j_k}) | x_0 = x \right] + \eta,$$

(A.17)
where \( d\alpha_k^j = \alpha_k^j - \alpha_k^j+1 \) (so \( \sum_{j=1}^{n_j} d\alpha_k^j = 1 \)). Note further that for \( x \geq \bar{z} \) and \( k = 1, 2, ..., n_j \),

\[
E \left[ e^{-r\tau_k^j} V \left( x_{\tau_k^j} \right) \bigg| x_0 = 0 \right] = E \left[ e^{-r\tau_k^j} E \left[ e^{-r(x-z)} \bigg| x_{\tau_k^j} \right] \bigg| x_0 = x \right] = E \left[ e^{-r\tau} (v - x) \bigg| x_0 = x \right] = V(x).
\]

Using this and the fact \( \sum_{j=1}^{n_j} d\alpha_k^j = 1 \) in (A.17) gives \( V(x) \leq L_j^j(x,0) < V_i(x) + \eta \) for all \( j > N \). Therefore, \( \lim_{m \to \infty} L_j^j(x,0) = V_i(x) \) for all \( x \geq \bar{z} \).

Consider next the case with \( x < \bar{z} \). Suppose by contradiction that there exists \( x < \bar{z} \) such that \( L_j^j(x,0) \to V_i(x) \) as \( j \to \infty \). Since \( L_j^j(x,0) \geq V_i(x) \) for all \( n \), there exists a subsequence \( \{j_r\} \), \( N \) and \( \gamma > 0 \) such that \( L_{j_r}^j(x,0) > V_i(x) + \gamma \) for all \( j_r > N \). Fix \( y \geq \bar{z} \) and let \( \tau_x = \inf \{t : x_t \leq x\} \). Since the seller can always delay trade until time \( \tau_x \), for all \( j_r > N \) it must be that

\[
L_{j_r}^j(y,0) \geq E[e^{-r\tau} L_{j_r}^j(x_{\tau_x}, 0) \bigg| x_0 = y] > E[e^{-r\tau} V_i(x_{\tau_x}) \bigg| x_0 = y] + E[e^{-r\tau} \gamma \bigg| x_0 = y] = E[e^{-r\tau} e^{-r(z-x)} (v - x) \bigg| x_{\tau_x}] \bigg| x_0 = y] + \left( \frac{y}{x} \right)^{\lambda} \gamma = V_i(y) + \left( \frac{y}{x} \right)^{\lambda} \gamma,
\]

which contradicts \( \lim_{j_r \to \infty} L_{j_r}^j(y,0) = V_i(y) \). Thus, \( \lim_{j \to \infty} L_j^j(x,0) = V_i(x) \) for all \( x < \bar{z} \).

Finally, I show that the limiting equilibrium outcome is efficient. Note that Lemma 13 implies that, for all \( i \in [0,1] \), the price \( P_j^j(x,i) \) that consumer \( i \) is willing to pay converges to \( V_i(x) + x \) for all \( x \leq z_{fji} \) as \( j \to \infty \). This in turn implies that, in the limit as \( j \to \infty \), the monopolist always sells at price \( V_i(x_t) + x_t \). By arguments similar to those in the proof of Lemma 1, for any \( v \in [v, \bar{v}] \), \( \tau_v = \inf \{t : x_t \leq \frac{x}{1-x} v\} \) solves

\[
\sup_x E \left[ e^{-r\tau} (v - V_i(x_{\tau_x}) - x_{\tau_x}) \bigg| x_0 = x \right].
\]

Since the monopolist always sells at a price of \( V_i(x_t) + x_t \) in the limit as \( j \to \infty \) and since consumers always make their purchase at the time that maximizes their surplus, in the limit a buyer with valuation \( v \) will make her purchase at the efficient time \( \tau_v \).


