Robust Comparative Statics in Large Dynamic Economies*

Daron Acemoglu† and Martin Kaae Jensen‡

April 20, 2012

PLEASE NOTE THAT THIS IS A VERY, VERY PRELIMINARY AND INCOMPLETE DRAFT. PLEASE DO NOT CIRCULATE.

Abstract

We consider infinite horizon economies populated by a continuum of agents who are subject to uninsurable idiosyncratic shocks. This framework contains models of saving and capital accumulation with incomplete markets in the spirit of works by Bewley, Aiyagari, and Huggett, and models of entry, exit and industry dynamics in the spirit of Hopenhayn's work as special cases. General and easy-to-apply comparative statics results are established with respect to exogenous parameters as well as various kinds of changes in the Markov processes governing the idiosyncratic shocks. A number of examples illustrate the usefulness of such results for macroeconomic modeling.

Keywords: Bewley-Aiyagari models, uninsurable idiosyncratic risk, infinite horizon economies, comparative statics.

JEL Classification Codes: C61, D90, E21.

*Comments and remarks are very welcome.
†Department of Economics, Massachusetts Institute of Technology (e-mail: daron@mit.edu)
‡Department of Economics, University of Birmingham. (e-mail: m.k.jensen@bham.ac.uk)
1 Introduction

In several settings, heterogeneous economic agents make dynamic choices with rewards determined by market prices or aggregate externalities. Those prices and externalities are in turn determined as the aggregates of the decisions of all agents in the market, and because there are sufficiently many agents, each ignores their impact on these aggregate variables. The equilibrium in general takes the form of a stationary distribution of decisions (or state variables such as assets), which remains invariant while each agent may experience changes in their decisions over time as a result of their type and stochastic shocks. Examples include: (1) Bewley-Aiyagari style models (e.g., Bewley (1986), Aiyagari (1994)) of capital accumulation in which each household is subject to idiosyncratic labor income shocks and make saving and consumption decisions taking future prices as given (or the related Huggett (1993) model where savings are in a zero net-supply risk-free asset). Prices are then determined as a function of the aggregate capital stock of the economy, resulting from all households’ saving decisions. (2) Models of industry equilibrium in the spirit of Hopenhayn (1992), where each firm has access to a stochastically-evolving production technology, and decides how much to produce and whether to exit given market prices, which are again determined as a function of total production in the economy. (3) Models with aggregate learning-by-doing externalities in the spirit of Arrow (1962) and Romer (1986), where potentially heterogeneous firms make production decisions, taking their future productivity as given, and aggregate productivity is determined as a function of total current or past production. (4) Search models in the spirit of Diamond (1982) where current production and search effort decisions depend on future thickness of the market.\footnote{Both models under (3) and (4) are typically set up without individual-level heterogeneity and with only limited stochastic shocks, thus stationary equilibria are often symmetric allocations. Our analysis covers significant generalizations of these papers where agents can be of different types and are subject to idiosyncratic shocks are represented by arbitrary Markov processes.}

Despite the common structure across these and several other models, little is known in terms of how the stationary equilibrium responds to a range of shocks including changes in preference and production parameters, and changes in the distribution of (idiosyncratic) shocks influencing each agent’s decisions. For example, even though the Bewley-Aiyagari model has become a workhorse in modern dynamic macroeconomics, most works rely on numerical analysis to characterize its implications.

In this paper, we provide a general framework for the study of large dynamic economies, nesting the above-mentioned models (or their generalizations) and show how robust comparative statics of stationary equilibria of these economies can be derived in a simple and tractable manner. Our first substantive theorem establishes monotonicity properties of fixed points of a class of mappings.
defined over general (non-lattice) spaces. This result is crucial for deriving comparative statics of stationary equilibria (which generally are not defined over spaces that are lattices).

Our second set of results use this theorem to show how the stationary equilibrium of large dynamic economies respond to a range of exogenous shocks affecting a subset or all economic agents. Examples include changes in the discount factor, changes in the borrowing limits, parameters of the utility function (e.g., the level of risk aversion), or parameters of the production function in the Bewley-Aiyagari model.

Our third set of results turn to an analysis of the implications of changes in the Markov processes guiding the behavior of stochastic shocks. Examples here include first-order stochastic dominance changes or mean-preserving spread of the stochastic processes affecting the labor incomes of households in Bewley-Aiyagari style models.

In each case, our results are both intuitive, easy to apply and “robust,” even though to the best of our knowledge no similar results have been derived for any of the specific models or for the general class of models under study here. By robustness, we mean that, like comparative static results in supermodular optimization problems and games, they obtain without requiring knowledge of specific functional forms and parameter values.

A noteworthy feature of our results is that in most cases, though how aggregates behave can be known robustly, very little or nothing can be said about individual behavior. Thus regularity of (market) aggregates is accompanied with irregularity of individual behavior. This highlights that our results are not a consequence of some implicit strong assumptions but follow because of the discipline that the market imposes on prices and aggregates.

This paper is related to two literatures. First, we are building on and extending a variety of well-known models of large dynamic economies, including Bewley (1986), Huggett (1993), Aiyagari (1994), Jovanovic (1982), Hopfenhayn (1992), Ericson and Pakes (1995). Though some of these papers contain certain specific results on how equilibria change with parameters (e.g., the effect of relaxing borrowing limits in Aiyagari (1994) and that of productivity on entry in Hopfenhayn (1992)), they do not present the general approach or the robust comparative static results provided here. In particular, to the best of our knowledge, none of these papers contain comparative statics either with respect to general changes in preferences and technology or with respect to changes in distributions of shocks, in particular mean-preserving spreads.

Second, our work is related to the robust competitive statics literature including Milgrom and Roberts (1994) and Milgrom and Shannon (1994). Selten (1970) and Corchón (1994) introduced and provided comparative statics for aggregative games where payoffs to individual agents depends on their strategies and an aggregate of others’ strategies. In Acemoglu and Jensen (2009), we
provided more general comparative static results for static aggregative games, thus extending the approach of Milgrom and Roberts (1994) to aggregative games (the earlier literature on aggregative games, including Corchón (1994), exclusively relied on the implicit function theorem). In Acemoglu and Jensen (2010), we considered large static environments in which payoffs depend on aggregates (and individuals ignored their impact on aggregates). To the best of our knowledge, the current paper is the first to provide general comparative statics results for dynamic economies.

We believe that the results provided here are significant for several reasons. First, as discussed at length by Milgrom and Roberts (1994), standard comparative statics methods such as those based on the implicit function theorem often run into difficulty unless there are strong parametric restrictions, and in the presence of such restrictions, the economic role of different ingredients of the model may be blurred. The existence of multiple equilibria is also a challenge to these standard approaches. Second, in fact most existing comparative results in the context of dynamic general equilibrium models, such as those studied here, rely not on comparative statics based on implicit function theorem but on numerical analysis, i.e., the model is solved numerically for two or several different sets of parameter values in order to obtain insights about how changes in parameters or policies will impact equilibrium in general (see, for example, Sargent and Ljungqvist’s (2004) textbook analysis of Bewley-Aiyagari and the related Huggett models). The results that follow from numerical analysis may be sensitive to parameter values and the existence of multiple equilibria, and they are particularly silent about the role of different assumptions of the model on the results. Our approach overcomes these difficulties by providing “robust” comparative static results for the entire set of equilibria and with a minimal amount of parametric restrictions. We believe that these problems increase the utility of our results and techniques, at the very least as a complement to existing methods of analysis in these dynamic models, since they also clarify the economic role of different ingredients of the model and typically indicate how these results can be extended to other environments.

The structure of the paper is as follows: Section 2 studies some applications. Section 3 describes the basic set-up including equilibrium and stationary equilibrium and also establishes existence under very general conditions. In Sections 4-5 we present our main comparative statics results. In Section 6 we return to the main examples from Section 2 and use our result to derive a variety of comparative statics results. Proofs are placed in Appendix I (Section 8.1). Appendix II (Section 8.2) contains a short summary of some results from stochastic dynamic programming used throughout the paper, and Appendix III (Section 8.3) discusses aggregation of risk through laws of large numbers.

---

2 This statement is valid even if there is a unique equilibrium for two different values of the exogenous parameter, see Milgrom and Roberts (1994) p.441-442, especially Figure 1.
2 Two Examples

This section describes two applications in detail, namely the Bewley-Aiyagari model of saving and capital accumulation, and Hopenhayn’s model of industry equilibrium. We also discuss how our large dynamic economies framework can be applied to models from growth theory and search equilibrium.

2.1 The Bewley-Aiyagari Model

Let $Q_t$ denote the aggregate capital-labor ratio at date $t$. Given a standard neo-classical production sector, $Q_t$ uniquely determines the wage $w_t = w(Q_t)$ and interest rate $r_t = r(Q_t)$ at date $t$ via the usual marginal product conditions of the firm. Household $i$ chooses their assets $x_{i,t}$ and consumption $c_{i,t}$ at each date in order to maximize:

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t v_i(c_{i,t}) \right]$$

subject to the constraint:

$$\tilde{\Gamma}_i(x_{i,t}, c_{i,t}, z_{i,t}, Q_t) = \{(x_{i,t+1}, c_{i,t+1}) \in [\tilde{b}_i, \tilde{b}_i] \times [0, \tilde{c}_i] : x_{i,t+1} \leq r(Q_t)x_{i,t} + w(Q_t)z_{i,t} - c_{i,t}\},$$

where $z_{i,t}$ denotes the labor endowment of household $i$, which is assumed to follow a Markov process, which can in principle vary across households. In addition, $\tilde{b}_i$ is an individual-specific lower bound on assets capturing both natural debt limits and other borrowing constraints; $\tilde{b}_i$ is an upper bound on assets introduced for expositional simplicity (it will not bind in equilibrium); and $\tilde{c}_i$ is an upper bound on consumption also introduced for expositional simplicity. The latter two ensure compactness and avoid unnecessary technical details, though it is worth noting that boundary/interiority/differentiability type assumptions will play no role in the results we present below.

More importantly, we assume throughout that there is no aggregate uncertainty, in the sense that total labor endowments in the economy is fixed, i.e.,

$$\int_{[0,1]} z_{i,t} di = 1,$$

where the mathematical meaning of this integral is discussed in Appendix III. Loosely, it can be interpreted as the “average” of the labor endowment of households in the economy.

---

3For example, the natural debt limit in the stationary equilibrium with rate of return $R$ on the assets would be $\tilde{b}_i = -\frac{R}{R+1} < 0$. 

4
Note that households in this economy are not assumed to be identical—they could differ with respect to their preferences and labor endowment processes. Assuming that \( v_i \) is increasing, we can substitute for \( c_{i,t} \) to get:

\[
E_0\left[ \sum_{t=0}^{\infty} \beta^t v_i(r(Q_t)x_{i,t} + w(Q_t)z_{i,t} - x_{i,t+1}) \right]
\]

It is convenient to define:

\[
u_i(x_i, y_i, z_i, Q, a_i) \equiv v_i(r(Q)x_i + w(Q)z_i - y_i) ,
\]

and

\[
\Gamma_i(x_i, z_i, Q) = \{ y_i \in [-\bar{b}_i, \bar{b}_i] : y_i \leq r(Q)x_i + w(Q)z_i \}.
\]

Finally, recalling that total labor endowment and the economy is equal to 1, the aggregate capital-labor ratio at date \( t \) is defined as

\[
Q_t = \int_{[0,1]} x_{i,t} di .
\] (1)

A stationary equilibrium will involve \( Q_t = Q^* \) for all \( t \), and thus will feature constant prices. Focusing on stationary equilibria, our general results establish results of the following form (in case there are more than one stationary equilibrium, the statements refer to the largest and smallest aggregates). Note that in this setting any increase in \( Q^* \) is also associated with an increase in output per capita:

- If agents become more patient, \( Q^* \) will increase. In particular, an increase in the discount rate \( \beta \) will lead to an increase in the steady state capital-labor ratio \( Q^* \) (Theorem 5).

- Any “positive shock” (to any subset of the agents not of measure zero) will lead to an increase in \( Q^* \) (Theorem 4). Positive shocks are defined formally in section 4.2, but interesting economic examples of positive shocks include:
  - A “tightening” of the borrowing constraints, \( \text{i.e.} \), a decrease in \( b_i \) (an increase in \( -b_i \)).
  - A decrease in marginal utilities, \( \text{i.e.} \), any increase in \( a_i \) when \( v_i = v_i(c_i, a_i) \) and
    \[
    D^2_{c_ia_i} u_i \leq 0 .
    \]
    For example, if \( a_i \) is the rate of absolute or relative risk aversion and
    \[
    D^2_{c_ia_i} u_i \leq 0 ,
    \]
    an increase in risk aversion will be a positive shock.

- Increases in “earnings risk” leads to an increase in \( Q^* \). Precisely, assume that \( u_i \) is strictly concave and exhibits HARA (Carroll and Kimball (1996)). For example if \( u_i \) exhibits a
Constant Rate of Relative Risk Aversion (CRRA) this will hold. Then any mean-preserving spread to (any subset of) the households’ stochastic processes will lead to an increase in $Q^*$ (Theorem 8).

In all of these cases, though the results are intuitive, we will also show that they cannot be derived from studying individual behavior, and in fact, while market aggregates respond robustly to these changes in the environment, very little or nothing can be said about the behavior of specific types of individuals.

2.2 Hopenhayn’s Model of Entry, Exit, and Firm Dynamics

Here we will study the model of Hopenhayn (1992). Hopenhayn’s model of entry, exit, and firm dynamics considers a continuum of firms $I$ subject to idiosyncratic productivity shocks with $z_{i,t} \in Z = [0, 1]$ denoting firm $i$’s shock at date $t$.

Upon entry, a firm’s productivity is drawn from a fixed probability distribution $\nu$, and from then on (as long as the firm remains active), its productivity follows a monotone Markov process with transition function $\Gamma(z, A)$.

Restrict attention to stationary equilibria where the sequence of (output) market prices is constant and equal to $p > 0$. Then at any point in time, the value of an active firm with productivity $z \in Z$ is determined by the value function $V$ which is the solution to the following functional equation:

$$V(p, z) = \max_{d \in \{0, 1\}, x \in \mathbb{R}^+} \left\{ (px - C(x, z) - c) + d\beta \int V(p, z') \Gamma(z, dz') \right\}$$

Here $C$ is the cost function and $c > 0$ a fixed cost paid each period by incumbent firms. $\beta$ is the discount rate, $x$ output, and $d$ a variable that captures active firms’ option to exit ($d = 1$ means that the firm remains active, $d = 0$ that it exits). $C$ is continuous, strictly decreasing in $z$, and strictly convex and increasing in $x$ with $\lim_{x \to \infty} C'(x, z) = \infty$ for all $z$. This ensures that there exists a unique function $V$ that satisfies this equation. Let $d^*(z, p)$ and $x^*(z, p)$ denote the optimal exit and output strategies for a firm with productivity $z$ facing the (stationary) price $p$. It is obvious that the firm will exit if and only if $\int V(p, z') \Gamma(z, dz') \geq 0$. Since $V$ will be strictly decreasing in $z$, this determines a unique (price-dependent) exit cut-off $\bar{z}_p \in Z$ such that $d^*(z, p) = 0$ if and only if $z < \bar{z}_p$.

Any firm that is inactive at date $t$ may enter after paying an entry cost $\gamma(M) > 0$ where $M$ is the measure of firms entering at that date, and $\gamma$ is a strictly increasing function. Given $p$

---

4So given the shock $z_{i,t}$ at date $t$, the probability of the shock laying in the set $A \subseteq Z$ at date $t + 1$ is $\Gamma(z_{i,t}, A)$. Monotonicity means that higher productivity at date $t$ makes higher productivity at date $t + 1$ more likely (mathematically $\Gamma(z', \cdot)$ first-order stochastically dominates $\Gamma(z, \cdot)$ whenever $z' \geq z$).

5This increasing cost of entry would result, for example, because there is a scarce factor necessary for entry (e.g.,
and the value function $V$ determined from $p$ as described above, new firms will consequently keep entering until their expected profits equals the entry cost:

$$
\int V(p, z') \nu(dz') - \gamma(M) = 0,
$$

(3)

where $\nu$ is the distribution of productivity for new entrants. Given $p$ (and from there $V$), this determines a unique measure of entrants $M_p$. Given $M_p$ and the above determined exit threshold $\bar{z}_p$, the stationary distribution of the productivities of active firms must satisfy:

$$
\mu_p(A) = \int_{z_i \geq \bar{z}_p} \Gamma(z_i, A) \mu_p(dz_i) + M \nu(A) \text{ all } A \in \mathcal{B}(Z)
$$

(4)

where $\mathcal{B}(Z)$ denotes the set of Borel subsets of $Z$.\(^6\)

The stationary equilibrium price level $p^*$ can now be determined as

$$
p^* = D[\int x^*(p^*, z_i) \mu_{p^*}(dz_i)],
$$

(5)

where $D$ is the inverse demand function for the product of this industry, which is assumed to be continuous and strictly decreasing. This equation makes it clear that the key aggregate (market) variable in this economy, the price level $p$, is determined as an aggregate of the stochastic outputs of a large set of firms. In consequence, from an economic point of view it is intuitive that the Hopenhayn model is a special case of our framework.

From a mathematical point of view, however, there is a slight difference between (5) and equation (1) which determined the market variables in the Aiyagari-Bewley model. Specifically, the right-hand side of (5) is not an integral of stochastic variables over a set of economic agents represented by the set $\mathcal{I}$ (or some subset of $[0,1]$). Nevertheless, this difference is of no consequence. To see this, we can proceed as follows: Instead of the random variable $x(p, z_i)$, which can be seen as a random variable defined across a set of heterogeneous firms, consider the random variable $\tilde{x}_i(p)$ drawn independently for a set of $N$ of firms from the distribution $\mu_p$ (on $(Z, \mathcal{B}(Z), \mu)$), where $N \subseteq \mathcal{I}$ is the set of active firms. Now let the distribution of productivities across the active firms at some date $t$ be denoted by $\eta_p : N \to Z$ (where this mapping potentially depends on $p$). Then the frequency distribution (image measure) is given by $\mu_p(A) = \eta_p\{i \in N : \eta_p(i) \in A\}$ where $A$ is any Borel subset of $Z$. Then

$$
\int_N \tilde{x}(\eta_p(i))di = \int_Z x^*(z, p) \mu_p(dz).
$$

land or managerial talent). Hopenhayn (1992) assumes that $\gamma(M)$ is independent of $M$. Our assumption simplifies the exposition, but it is not critical for our results.

\(^6\) Hopenhayn (1992) refers to the measure $\mu_p$ as the state of the industry.
In words, the expected output of the “average” active firm equals the integral of $x^*(\cdot, p)$ under the measure $\mu_p$. Therefore, (5) can be equivalently written as

$$p = H((\tilde{x}_i(p))_{i \in \mathcal{I}}) \equiv D\left(\int_{i \in \mathcal{I}} \tilde{x}_i(p) \, di\right) = D\left[\int_{i \in \mathcal{N}} x(p^*, z_i) \mu_p^*(dz_i)\right],$$

which now has exactly the same mathematical form as (1), making it transparent how the Hoppenhayn model is a special case of our framework.

In this setting, our general results will lead to the following comparative static results for market aggregates in stationary equilibria:

- A reduction in the fixed cost of operation $c$ or an increase in the transition function $\Gamma$ will increase aggregate output and lower equilibrium price.
- A first-order stochastically dominant shift in the entrants’ productivity distribution $\nu$ will increase aggregate output and lower the equilibrium price.
- Positive shocks to profit functions, i.e., changes in parameters that increase the desired level of production at a given price, will increase aggregate output and lower the equivalent price.

### 2.3 Additional examples

Several other models can also be cast as special cases of the framework presented here, enabling ready applications of the comparative static results developed below. We describe these models briefly in this subsection since, to economize on space and avoid repetition, we will not explicitly show how our results can be applied for these models.

1. A variety of models where a large number of firms or economic actors create an aggregate externality on others would also be a special case of our framework. A well-known example of this class is Romer’s paper on endogenous growth where the aggregate capital stock of the economy determines the productivity of each firm (Romer (1986)). Though Romer’s model was deterministic and did not feature any heterogeneity across firms, one could consider generalizations where such stochastic elements are important. For example, we can consider a continuum $\mathcal{I}$ of firms each with production function for a homogeneous final good given by

$$y_{i,t} = f(k_{i,t}, A_{i,t}Q_t)$$
where $f$ exhibits diminishing returns to scale, is increasing in both of its arguments, and $A_{i,t}$ is independent across producers and follows a Markov process (which can again vary across firms).\footnote{In Romer’s model $f$ exhibits constant returns to scale, which can also be allowed here, but in that case the relevant comparative statics are on the growth rate of $Q_t$.} Each firm faces an exogenous cost of capital $R$. Romer (1986) considered an externality operating from current capital stocks, so that

$$Q_t = \int k_{i,t} di.$$ 

One could also consider “learning by doing” type externalities that are a function of past cumulative output, i.e.,

$$Q_t = \sum_{\tau=t-T-1}^{t-1} \int y_{i,\tau} di,$$

for some $T < \infty$. Under these assumptions, all of the results derived below can be applied to this model.

2. Search models in the spirit of Diamond (1982) and Mortensen (1982), where members of a single population match pairwise to form productive relationships, also constitute special case of this framework. In Diamond’s (1982) model, for example, individuals first make costly investments in order to produce (“collective a coconut”) and then search for others who have also done so to form trading relationships. The aggregate variable, taken as given by each agent, is the fraction of agents that are searching for partners. This determines matching probabilities and thus the optimal strategies of each agent. Thus various generalizations of Diamond’s model, or for that matter other search models, can also be studied using the framework presented below.

One relevant example in this context is Acemoglu and Shimer (2000), which combines elements from directed search models of Moen (1997) and Acemoglu and Shimer (1999) together with Bewley-Aiyagari style models. In this environment, each individual decides whether to apply to high wage or low wage jobs, recognizing that high wage jobs will have more applicants and thus lower offer rates (these offer rates and exact wages are determined in equilibrium is a function of applications decisions of agents). Individuals have concave preferences and do not have access to outside insurance opportunities, so use their own savings to smooth consumption. Unemployed workers with limited assets then prefer to apply to low wage jobs. Acemoglu and Shimer (2000) assumed the fixed interest rate and used numerical methods to give a glimpse of the structure of equilibrium and to argue that high unemployment benefits can increase output by encouraging more workers apply to high
wage jobs. This model—and in fact a version with an endogenous interest-rate—can also be cast as a special case of our framework and thus, in addition to basic existence results, a range of comparative static results can be obtained readily.

3 Large Dynamic Economies

This section describes the general setup of large dynamic economies, which can be viewed as a generalization of Aiyagari (1994) and Bewley (1986) to include heterogeneous agents and general stochastic processes. The crucial feature, as in these works, is the absence of “aggregate risk” as captured by the fact that all interaction between the agents takes place through a one-dimensional deterministic market aggregate (typically, aggregate capital or a price variable). The section also deals briefly with the way in which idiosyncratic risk is eliminated at the aggregate level, a question related to various versions of the Law of Large numbers and dealt with in detail in Section 8.3.

3.1 Preferences and Technology

The setting of our analysis is infinite horizon, discrete time economies populated by a continuum of agents $I = [0, 1]$.

Each agent $i \in [0, 1]$ is subject to ( uninsurable) idiosyncratic shocks in the form of a Markov process $(z_{i,t})_{t=0}^{\infty}$ where $z_{i,t} \in Z_i \subseteq \mathbb{R}^M$. The only assumption we place on $(z_{i,t})_{t=0}^{\infty}$ is that it must have a unique invariant distribution $\mu_{z_i}$. A special case of this is when the $z_{i,t}$’s are i.i.d. in which case $z_{i,t}$ has the distribution $\mu_{z_i}$ for all $t$.

Agent $i$’s action set is $X_i \subseteq \mathbb{R}^n$, and the agent’s objective is, for given initial conditions $(x_{i,0}, z_{i,0}) \in X_i \times Z_i$ to solve:

$$\sup \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t u_i (x_{i,t}, x_{i,t+1}, z_{i,t}, Q_t, a_i) \right]$$

s.t. $x_{i,t+1} \in \Gamma_i (x_{i,t}, z_{i,t}, Q_t, a_i), \ t = 0, 1, 2, \ldots$ (6)

Ignoring for a moment $Q = (Q_0, Q_1, Q_2, \ldots), Q_t \in \mathbb{Q} \subseteq \mathbb{R}$, (6) is a standard dynamic programming problem as treated at length in Stokey and Lucas (1989). $u_i : X_i \times X_i \times Z_i \times \mathbb{Q} \times A_i \rightarrow \mathbb{R}$ is the \textit{instant payoff function}, $\Gamma_i : X_i \times Z_i \times \mathbb{Q} \times A_i \rightarrow 2^{X_i}$ is the \textit{constraint correspondence}, $\beta$ is the agents’ common discount factor, and $a_i \in A_i \subseteq \mathbb{R}^P$ is a vector of parameters with respect to which we wish to do comparative statics. In this setting, a \textit{strategy} $x_i = (x_{i,1}, x_{i,2}, \ldots)$ is a

---

8 Throughout, all sets are equipped with the Lebesgue measure and Borel algebra (and products of sets with the product measure and product algebra). For a set $Z$, the Borel algebra is denoted by $\mathcal{B}(Z)$ and the set of probability measures on $(Z, \mathcal{B}(Z))$ is denoted by $\mathcal{P}(Z)$.

Although we consider for simplicity only $I = [0, 1]$, our results hold for any non-atomic measure space of agents. This includes a setting such as that of Al-Najjar (2004), where the set of agents is countable and the measure is finitely additive (see section 8.3 in the appendix for further details). In fact, our comparative statics results remain valid for economies with a finite set of agents, provided that appropriate assumptions are made to ensure the absence of aggregate risk and existence of equilibrium (here we have not imposed concavity/convexity type assumptions since the continuum plays a “convexifying” role).
sequence of random variables defined on the histories of shocks, i.e., a sequence of (measurable) maps \( x_{i,t} : Z_{i,t}^{-1} \rightarrow X_i \) where \( Z_{i,t}^{-1} = \prod_{\tau=0}^{t-1} Z_i \). A feasible strategy is one that satisfies the constraints in (6), and an optimal strategy is a solution to (6). When a strategy is optimal, it is denoted by \( x_i^* \). The following standard assumptions will ensure the existence of an optimal strategy given any choice of \( Q, \mathbf{a}, \) and \((x_{i,0}, z_{i,0})\):

**Assumption 1** \( \beta \in (0, 1) \) and for all \( i \in \mathcal{I} \): \( X_i \) is compact, \( u_i \) is bounded and continuous, and \( \Gamma_i \) is non-empty, compact-valued, and continuous.

Note that optimal strategies are not necessarily unique under Assumption 1 (in particular, no convexity/quasi-concavity assumptions are imposed).

Let us next turn to the sequence of market aggregates \( Q = (Q_0, Q_1, Q_2, \ldots) \). Each \( Q_t \) is a deterministic real variable, and all market interaction takes place through these aggregates. So in our setting there is no aggregate uncertainty (for a detailed discussion of this feature see e.g. Lucas (1980), Bewley (1986), and Aiyagari (1994)). To get some understanding of the definition to follow, imagine we are in the income-fluctuation setting of Aiyagari (1994) and that \( Q_t \) is the capital-labor ratio at date \( t \). Profit maximization of a standard neo-classical production technology entails \( r_t = r(Q_t) = f'(Q_t) \) and \( w_t = w(Q_t) = f(Q_t) - f'(Q_t)Q_t \) where \( f \) is the intensive production technology, \( r_t \) is the interest rate, and \( w_t \) the real wage at date \( t \). So at any date \( t \), the constraint of agent \( i \) depends only on \( Q_t \) by way of the interest and wage rates: 
\[
\Gamma_i(x_{i,t}, z_{i,t}, Q_t) = \{ y_i \in [-\tilde{b}_i, \tilde{b}_i] : y_{i,t} \leq r(Q_t)x_{i,t} + w(Q_t)z_{i,t} \}.
\]

Note that one way in which exogenous parameters could enter into this model would be through the production technology (having \( f = f(Q_t, \tilde{a}) \) where then \( a_i = \tilde{a} \) for all \( i \)).

### 3.2 Markets and Aggregates

Having now explained the economic intuition behind the market aggregates, we can explain how they are determined. Recall that in the income-fluctuation setting just described, \( Q_t \) was the capital-labor ratio at date \( t \). Since \( x_{i,t} \) is savings of individual \( i \) at date \( t \), clearing of the capital markets therefore implies that \( Q_t = H((x_{i,t})_{i \in \mathcal{I}}) \), where \( H((x_{i,t})_{i \in \mathcal{I}}) \) is the mean of the strategies (the savings-labor ratio):\(^{10}\)

\[
H((x_{i,t})_{i \in \mathcal{I}}) = \int_{[0,1]} x_{i,t} \, di
\]  

\(^9\)Economically, the map \( x_{i,t} \) is a state-dependent contingency plan: Given a realized history of shocks \( z_{i,t}^{-1} \in Z_{i,t}^{-1} \) the agent will choose \( x_{i,t} = x_{i,t}(z_{i,t}^{-1}) \) at date \( t \).

\(^{10}\)When \( X_i \) is multi-dimensional, we would usually generalize this by taking \( H((x_{i,t})_{i \in \mathcal{I}}) = M(\int_{[0,1]} x_{1,t} \, di, \ldots, \int_{[0,1]} x_{N,t} \, di) \) where \( M : \mathbb{R}^N \rightarrow \mathbb{R} \) is a continuous and coordinatewise increasing function.
For applications, (7) is by far the most important example of a so-called *aggregator* as defined next (see also Acemoglu and Jensen (2009, 2010)). For this paper’s results, we allow $H$ to be a general function as long as certain technical conditions are satisfied. Since this unavoidably gets somewhat abstract, the reader may in a first reading take the functional form (7) as given and skip directly to the definition of an equilibrium (Definition 2).

A real-valued function is said to be *convexifying* if it maps any non-empty subset into a convex subset of $\mathbb{R}$. A function $H$ that maps a vector of random variables $(\tilde{x}_i)_{i \in \mathcal{I}}$ into a real number is said to be *increasing* if it is increasing in the first-order stochastic dominance order $\succeq_{st}$, i.e., if $H((\tilde{x}_i)_{i \in \mathcal{I}}) \geq H((x_i)_{i \in \mathcal{I}})$ whenever $\tilde{x}_i \succeq_{st} x_i$ for all $i \in \mathcal{I}$. Finally, any topological statement relating to sets of random variables (probability distributions), refer to the weak $*$-topology (see e.g. Stokey and Lucas (1989), Hopenhayn and Prescott (1992)).

**Definition 1 (Aggregator)** An aggregator is a continuous, increasing, and convexifying function $H$ that maps the agents’ strategies at date $t$ into a real number $Q_t \in Q$ (with $Q \subseteq \mathbb{R}$ denoting the range of $H$). The value,

$$Q_t = H((x_{i,t})_{i \in \mathcal{I}}),$$

is referred to as the (market) aggregate at date $t$.

**Remark 1** Note that if $H$ is an aggregator, then so is any continuous and increasing transformation of $H$. Thus (7) represents, up to a monotone transformation, the class of separable functions which is consequently a special case of this paper’s aggregation concept (see for example Acemoglu and Jensen (2009) for a detailed discussion of separable aggregators).

The reader who is familiar with the extensive literature on aggregation of uncertainty by way of some form of the law of large numbers will recognize that the conditions in Definition 1 will naturally be satisfied for any reasonable aggregation procedure. This does not mean that there is no ambiguity, however. In fact, even with a functional form such as (7), there is no universal agreement on how this expression should be defined. To avoid unnecessary technical discussion at this point, we have relegated a detailed discussion of this issue to Appendix III. Because we have simply defined an aggregator as a real-valued function, we are in effect side-stepping this issue here which has the benefit of not committing us to any specific way of integrating across random variables. In particular, our results’ validity are not affected by the controversy surrounding aggregation of risk and the law of large numbers.\(^{12}\)

---

\(^{11}\)Let $\tilde{x}_i$ and $x_i$ be a random variables on a set $X_i$ with distributions $\tilde{\mu}_{x_i}$ and $\mu_{x_i}$. Then $\tilde{x}_i$ first-order stochastically dominates $x_i$, written $\tilde{x}_i \succeq_{st} x_i$, if \(\int_{X_i} f(x_i) \tilde{\mu}_{x_i}(dx_i) \geq \int_{X_i} f(x_i) \mu_{x_i}(dx_i)\) for any increasing function $f : X_i \rightarrow \mathbb{R}$.

\(^{12}\)Our results are robust to all the standard specifications within this literature, including non-standard set-ups.
3.3 Equilibrium

Definition 2 (Equilibrium) Let the initial conditions \((z_{i,0}, x_{i,0})\) as well as the exogenous variables \((a_i)\) be given. An equilibrium \(\{Q^*, (x^*_i)\}_{i \in I}\) is a sequence of market aggregates and a sequence of strategies for each of the agents such that:

1. For each agent \(i \in I\), \(x^*_i = (x^*_{i,1}, x^*_{i,2}, x^*_{i,3}, \ldots)\) solves (6) given \(Q^* = (Q^*_0, Q^*_1, Q^*_2, \ldots)\) and the initial conditions \((z_{i,0}, x_{i,0})\).

2. The market aggregate clears at each date, i.e., \(Q^*_t = H((x^*_i, t)_{i \in I})\) for all \(t = 0, 1, 2, \ldots\).

Theorem 1 (Existence of Equilibrium) Under Assumption 1, there exists an equilibrium for any choice of initial conditions \((z_{i,0}, x_{i,0})\) and any choice of exogenous variables \((a_i)\).

As with all other results, the proof of Theorem 1 is presented in Appendix I.

Remark 2 Note that the convexifying property of aggregators plays a critical role in the above proof because we have not placed any concavity assumptions on \(u_i\) or convexity assumptions on \(\Gamma_i\). If we do assume that each \(u_i\) is concave in the strategies and that \(\Gamma_i\) has a convex graph, then there will exist an equilibrium even if \(H\) is not convexifying.

3.4 Stationary Equilibria

Most of our results will be about stationary equilibria. The simplest way to define a stationary equilibrium in stochastic dynamic settings involves assuming that the initial conditions \((x_{i,0}, z_{i,0})\) are random variables. To simplify notation we use the symbol \(\sim\) to express that two random variables have the same distribution.

Definition 3 (Stationary Equilibrium) Let the exogenous variables \((a_i)\) be given. A stationary equilibrium \(\{Q^*, (x^*_i)\}_{i \in I}\) is a market aggregate and a strategy for each of the agents such that:

1. For each agent \(i \in I\), \(x^*_i = (x^*_{i,1}, x^*_{i,2}, x^*_{i,3}, \ldots)\) solves (6) given \(Q^* = (Q^*_0, Q^*_1, Q^*_2, \ldots)\), the stationary process \(z_{i,t} \sim z_i\) all \(t\), and the randomly drawn initial conditions \((x_{i,0}, z_{i,0}) \sim x^*_i \times z_i\).

\(\text{such as that of Al-Najjar (2004) (see also footnote 8).}\)

One technical detail that may be worth noting is that thing that \(Q_t\) determined through an aggregator as above may in some cases not be a real (deterministic) variable but rather a degenerate random variable placing probability 1 on some number \(Q_t\). Since agents maximize their expected payoffs, this distinction is of no importance.

\(\text{13} \text{Obviously, the probability distribution of } z_i \text{ is } \mu_{z_i} \text{ (the invariant distribution of the Markov process governing } z_{i,t}).\)
2. The market aggregates clear (at all dates), i.e.,

\[ Q^* = H((x^*_i)_{i \in I}) \]

A market aggregate \( Q^* \) of a stationary equilibrium will be referred to as an equilibrium aggregate and the set of equilibrium aggregates given \( a = (a_i)_{i \in I} \) denoted by \( E(a) \). The smallest and largest element in \( E(a) \) are referred to as the smallest and largest equilibrium aggregates, respectively.

While Assumption 1 implies existence of an equilibrium, it does not imply existence of a stationary equilibrium. In fact, it does not even ensure that the individual agents’ decision problems will admit a stationary strategy given a stationary sequence of market aggregates. Note that in a stationary equilibrium, agent \( i \) faces a stationary sequence of aggregates \((Q, Q, \ldots)\) and the stationary risk process \( z_{i,t} \sim z_i \) (with distribution \( \mu_{z_i} \)). The agent consequently faces a stationary dynamic programming problem whose value function \( v_i \) is determined by the following functional equation:

\[
v_i(x_i, z_i, Q, a_i) = \sup_{y_i \in \Gamma_i(x_i, z_i, Q, a_i)} [u_i(x_i, y_i, z_i, Q, a_i) + \beta \int v_i(x_i, z_i', Q, a_i) \mu_{z_i}(dz_i')] \tag{9}
\]

Given \( v_i \), we can then compute the (stationary) policy correspondence:

\[
G_i(x_i, z_i, Q, a_i) = \arg \sup_{y_i \in \Gamma_i(x_i, z_i, Q, a_i)} [u_i(x_i, y_i, z_i, Q, a_i) + \beta \int v_i(x_i, z_i', Q, a_i) \mu_{z_i}(dz_i')] \tag{10}
\]

The stationary strategy \( x^*_i = (x^*_1, x^*_2, \ldots) \) of Definition 3 is a sequence of random variables such that the distribution of \( x^*_i \) is an invariant distribution for this stationary decision problem. The next assumption will ensure that such a stationary optimal strategy exists given any stationary sequence of market aggregates. To simplify notation we typically write \( u_i(x_i, y_i, z_i, Q, a_i) \) in place of \( u_i(x_{i,t}, x_{i,t+1}, z_{i,t}, Q_t) \), and similarly we write \( \Gamma_i(x_i, z_i, Q, a_i) \) for the constraint correspondence.

**Assumption 2** \( X_i \) is a lattice, and given any choice of \( z_i, Q, \) and \( a_i: u_i(x_i, y_i, z_i, Q, a_i) \) is supermodular in \((x_i, y_i)\) and the graph of \( \Gamma_i(\cdot, z_i, Q, a_i) \) is a sublattice of \( X_i \times X_i \).

**Remark 3** Fixing and suppressing \((z_i, Q, a_i)\), \( \Gamma_i \)'s graph is a sublattice of \( X_i \times X_i \), if for all \( x^1_i, x^2_i \in X_i \) and \( y^1_i \in \Gamma_i(x^1_i) \) and \( y^2_i \in \Gamma_i(x^2_i) \) imply that \( y^1_i \land y^2_i \in \Gamma_i(x^1_i \land x^2_i) \) and \( y^1_i \lor y^2_i \in \Gamma_i(x^1_i \lor x^2_i) \). When \( X_i \subseteq R \) (one-dimensional choice sets), this will hold if and only if the correspondence is ascending in \( x_i \) (Topkis (1978)), meaning that for all \( x^2_i \geq x^1_i \) in \( X_i \), \( y^1_i \in \Gamma_i(x^1_i) \) and \( y^2_i \in \Gamma_i(x^2_i) \) imply that \( y^1_i \land y^2_i \in \Gamma_i(x^1_i) \) and \( y^1_i \lor y^2_i \in \Gamma_i(x^2_i) \).

\(^{14}\)A solution \( v_i \) exists and is unique under assumption 1, see Stokey and Lucas (1989).
Assumption 2 implies that the policy correspondence of the agent $G_i(x_i, z_i, Q, a_i)$ is ascending in $x_{i,t}$ (defined formally in the previous remark).\textsuperscript{15} Precisely, for $x_{i,t}^2 \geq x_{i,t}^1$ and $y_{i,t}^1 \in G_i(x_{i,t}^1, z_i, Q, a_i)$, $j = 1, 2$, we have $y_{i,t}^1 \wedge y_{i,t}^2 \in G_i(x_{i,t}^1, z_i, Q, a_i)$ and $y_{i,t}^1 \vee y_{i,t}^2 \in G_i(x_{i,t}^2, z_i, Q, a_i)$.\textsuperscript{16} In dynamic economies, this is typically a rather weak requirement (as opposed to assuming that $G_i$ is ascending in $Q_t$ which is highly restrictive). For example, in the Bewley-Aiyagari model (Aiyagari (1994)), the condition on $\Gamma_i$ is trivially satisfied and we have $u_i(x_i, y_i, z_i, Q, a_i) = \tilde{u}_i(r(Q)x_i + w(Q)z_i - y_i)$ where $\tilde{u}_i(c_i)$ is agent $i$’s utility from the consumption $c_i$; this implies that $u_i$ will be supermodular in $(x_i, y_i)$ if and only if the utility function $\tilde{u}_i$ is concave.\textsuperscript{17}

**Theorem 2 (Existence of Stationary Equilibrium)** Suppose Assumptions 1 and 2 hold. Then there exists a stationary equilibrium and the set of equilibrium aggregates is compact. In particular, there always exist a smallest and a largest equilibrium aggregate.

### 4 Main Results I: Changes in Exogenous Variables

In this section, we start with a new technical result that establishes monotonicity properties of fixed points under sufficiently general conditions to be used in the context of the analysis of comparative statics in large dynamic economies. We then use this result to derive three general comparative statics results that predict how the set of stationary equilibrium aggregates (Definition 3) changes in response to a change in various exogenous parameters. The first of these derives the effects of changes in the exogenous parameters $a = (a_i)_{i \in I}$ on the smallest and largest equilibrium aggregates. The second result shows how a change in the discount factor (“the level of patience”) affects these equilibrium aggregates. Our third tracks the effect of such changes on individual strategies but in order to prove such a result much more restrictive assumptions are needed.

For our results on the aggregates, what is most striking is that we do not need to assume anything about how the sequence of market variables ($Q_0, Q_1, Q_2, \ldots$) enters into the payoff functions and constraint correspondences (aside from continuity, cf. Assumption 1).\textsuperscript{18} So our assumptions

\textsuperscript{15}An ascending mapping is the same as a mapping that is increasing in the strong set order, see for example Milgrom and Shannon (1994).

\textsuperscript{16}See Hopenhayn and Prescott (1992), Proposition 2 for a proof of this claim (Hopenhayn and Prescott consider a slightly more general situation and also ensure that $G_i$ will be ascending in $z_i$ which of course requires additional assumptions. Nonetheless, one easily sees that their proof implies that $G_i$ will be ascending in $x_i$ under assumption 2).\textsuperscript{2)

\textsuperscript{17}This is easiest to see in the twice differentiable case: Since $D^2_{x_iy_i} u_i = -r(Q)\tilde{u}_i''$, $D^2_{x_iy_i} u_i \geq 0$ (supermodularity) holds if and only if $\tilde{u}_i'' \leq 0$ (concavity).

\textsuperscript{18}It is useful to note that our results are valid for a finite number of agents as long as these all take the market aggregates as given. This reiterates that our results are not “aggregation” results that depend on the continuum assumption.
do not restrict us to “monotone economies” (see e.g. Mirman et al (2008)). Because of this, we can not in general say anything about how the individual strategies will respond to changes in exogenous parameters. Indeed, individual strategies’ response will in general be highly irregular – unless we add additional monotonicity assumptions as indeed we are forced to do for our individual comparative statics result. But at the market level, the irregularity of individual behavior is nonetheless restricted at the market level so as to lead to considerable regularity in the aggregate.\footnote{Note here that when doing comparative statics in general equilibrium one would be inclined to first try to pin down the individual responses and then aggregate. What the previous discussion shows is that this is not a good idea, in fact it will not work simply because individual responses are not in general well-behaved. The regular comparative statics results we identify below are a feature of the market level and equilibrium forces impacting aggregate variables.}

4.1 Monotonicity of Fixed Points

At the heart of our substantive results is a theorem that enables us to establish monotonicity of fixed points defined over general (non-lattice) spaces. We start with this theorem.

Comparative statics of equilibria (whether they are represented by vectors or distributions) boils down to studying the behavior of the fixed points of some mapping $F : X \times T \rightarrow 2^X$ where $x \in X$ is the variable of interest. In most of our applications, $x$ is a probability distribution, and $t \in T$ denotes exogenous variables with respect to which comparative statics will be conducted. Mathematically, the question is how the set of fixed points

$$\Lambda(t) \equiv \{ x \in X : x \in F(x, t) \}$$

varies with $t$.

The technical problem one faces is that when $X$ is a set of probability measures, it is generally not a lattice in any natural order (Hopenhayn and Prescott (1992), p.1389).\footnote{Further, for general equilibrium analysis, one cannot work with increasing selections from $F$, making it necessary to study the set-valued case in general. In general, increasing selections may not exist in the setting of the present paper, but more importantly, even if they were to exist, general equilibrium analysis requires all invariant distributions to be taken into account. This makes it impossible to use such a result as Corollary 3 in Hopenhayn and Prescott (1992) which concerns (single-valued) increasing functions.} Theorem 3, the main result of this subsection, enables us to derive monotonicity results in this case.

When sets are not lattices, we cannot require that the mapping $F$ is ascending (increasing in the strong set order). Instead we use here a monotonicity notion due to Smithson (1971):

Definition 4 (Type I and Type II Monotonicity (Smithson (1971))) Let $X$ and $Y$ be ordered sets. A correspondence $F : X \rightarrow 2^Y$ is:

1. Type I monotone if for all $x_1 \preceq x_2$ and $y_1 \in F(x_1)$, there exists $y_2 \in F(x_2)$ such that $y_1 \preceq y_2$. 

2. Type II monotone if for all \( x_1 \preceq x_2 \) and \( y_2 \in F(x_2) \), there exists \( y_1 \in F(x_1) \) such that \( y_1 \preceq y_2 \).

When a correspondence \( F \) is defined on a product set, \( F : X \times T \to 2^Y \), where \( T \) is also a partially ordered set, we say that \( F \) is type I (type II) monotone in \( t \), if \( F : \{x\} \times T \to 2^Y \) is type I (type II) monotone for each \( x \in X \). Type I/II monotonicity in \( t \) is defined similarly by keeping \( t \) fixed. If \( F : X \times T \to 2^Y \) is type I (type II) monotone in \( x \) as well as in \( t \), we simply say that \( F \) is type I (type II) monotone.

Note that for a correspondence \( F \) to be type I or type II monotone, no specific order structure for the values or domain of \( F \) is required. This is in sharp contrast to such concepts as monotonicity with respect to the weak or strong set orders (see e.g. Shannon (1995)).

The main result, upon which all of the rest of our results built, is

**Theorem 3 (Comparing Equilibria)** Let \( X \) be a compact topological space equipped with a closed order \( \succeq \), \( T \) a partially ordered set, and let \( F : X \times \{t\} \to 2^X \) be upper hemi-continuous for each \( t \in T \). Define the (possibly empty-valued) fixed point correspondence \( \Lambda(t) = \{x \in X : x \in F(x,t)\} \), \( \Lambda : T \to 2^X \cup \emptyset \). Then if \( F \) is type I monotone, so is \( \Lambda \); and if \( F \) is type II monotone, so is \( \Lambda \).

A natural corollary of this theorem is also useful:

**Corollary 1** Let \( \Lambda(t) \subset X \) be the fixed point set of Theorem 3 (for given \( t \in T \)), assumed here to be non-empty \( \Lambda(t) \neq \emptyset \) for \( t \in T \), and consider a continuous and monotone function \( H : X \to \mathbb{R} \). Define the greatest and least selections from \( H \circ \Lambda(t) : \tilde{\Lambda}(t) = \sup_{x \in \Lambda(t)} H(x) \) and \( \underline{h}(t) = \inf_{x \in \Lambda(t)} H(x) \). Then if \( \Lambda(t) \) is type I monotone, \( \tilde{\Lambda}(t) \) will be monotone non-decreasing; and if \( \Lambda(t) \) is type II monotone, \( \underline{h}(t) \) will be monotone non-decreasing.

### 4.2 Comparative Statics of Equilibrium Aggregates

Consider the instant utility function \( u_i = u_i(x_i, y_i, z_i, Q, a_i) \) of an agent \( i \in I \). \( u_i \) will exhibit increasing differences in \( y_i \) and \( a_i \) if \( u_i(x_i, y_i^2, z_i, Q, a_i) - u_i(x_i, y_i^1, z_i, Q, a_i) \) is non-decreasing in \( a_i \) whenever \( y_i^2 \geq y_i^1 \). If \( X_i, A_i \subset \mathbb{R} \) and \( u_i \) is differentiable, increasing differences in \( y_i \) and \( a_i \) is equivalent to having \( D^2_{y_ia_i} u_i \geq 0 \). Increasing differences is of course a very well-known condition in comparative statics analysis (see for example Topkis (1998)).

Next consider the constraint correspondence \( \Gamma_i(x_i, z_i, Q, a_i) \) of agent \( i \). Following Hopenhayn and Prescott (1992), \( \Gamma_i \) is said to have strict complementarities in \( (x_i, a_i) \) if for any fixed choice of \( (z_i, Q) \) it holds for all \( x_i^2 \geq x_i^1 \) and \( a_i^2 \geq a_i^1 \), that \( y \in \Gamma(x_i^1, z_i, Q, a_i^1) \) and \( \tilde{y} \in \Gamma(x_i^2, z_i, Q, a_i^1) \) implies...
As an illustration consider the constraint correspondence of the Bewley-Aiyagari model of section 2.1,

\[ \Gamma_i(x_i, z_i, Q, a_i) = \{ y_i \in [a_i, b_i] : y_i \leq r(Q)x_i + w(Q)z_i \} \]

where we have treated the borrowing limit \(-b_i\) as the parameter (so \(a_i = -b_i\) where \(b_i\) is the agents’ borrowing limit). Since it is clear that when \(x_i^2 \geq x_i^1\) and \(a_i^2 \geq a_i^1\), \(y \land \tilde{y} \in [a_i^1, r(Q)x_i^1 + w(Q)z_i]\) and \(\tilde{y} \in [a_i^1, r(Q)x_i^2 + w(Q)z_i]\) imply that 
\[ y \land \tilde{y} = \min\{y, \tilde{y}\} \in [a_i^1, r(Q)x_i^1 + w(Q)z_i] \] and \(y \lor \tilde{y} = \max\{y, \tilde{y}\} \in [a_i^2, r(Q)x_i^2 + w(Q)z_i]\), this correspondence has strict complementarities in \((x_i, a_i)\).

Hence a “tightening” of the borrowing limits in a Bewley-Aiyagari economy will be a positive shock according to the following definition (note that since \(a_i\) does not affect the utility function in this case, the increasing differences part is trivially satisfied):

**Definition 5 (Positive Shocks)** Consider an agent \(i \in I\). A (coordinatewise) increase in the exogenous parameters \(a_i\) is a positive shock if \(u_i(x_i, y_i, z_i, Q, a_i)\) exhibits increasing differences in \(y_i\) and \(a_i\), and \(\Gamma_i(x_i, z_i, Q, a_i)\) has strict complementarities in \((x_i, a_i)\).

Definition 5 gives the “correct” notion of a positive shock: If an increase in \(a_i\) is a positive shock then the policy correspondence \(G_i(x_i, z_i, Q, a_i)\) defined in (10) will be ascending in \(a_i\) whenever assumption 2 holds. Having a determinate impact of the change in the environment, here captured by \(a_i\), on individual decision problem, i.e., for given aggregates and prices, is clearly a prerequisite for meaningful equilibrium comparative statics. Hence we follow Acemoglu and Jensen (2009) in presenting a positive shock as a definition rather than stating the conditions involved as an assumption.

Importantly, however, this individual behavior conclusion is very different from the (general) equilibrium results which are our focus. Positive shocks to the \(a_i\)'s of a subset of players will lead to increases in those players’ strategies for fixed market aggregates. But in (general) equilibrium, the market variables will also change—in particular, the initial change strategies will change the equilibrium aggregates which will lead to further changes in everyone’s strategies, further changes in equilibrium aggregates, and so on until a new equilibrium is reached. Since we have assumed essentially nothing—except continuity—about how the market aggregates enter into the agents’ decision problems, it may at first appear that very little can be said about how aggregates will change. But as this section’s main result shows, on the contrary, we can determine how market aggregates behave quite precisely:

\[ \text{Having strict complementarities is a weaker condition than assuming that the graph of } \Gamma_i \text{ is a sublattice (of } X_i \times X_i \times A_i \text{ for given } (z_i, Q)\). See Hopenhayn and Prescott (1992) for further details and discussion. \]
**Theorem 4 (Comparative Statics of the Aggregate)** Under Assumptions 1 and 2, a positive shock to $a_i$ (for all players or any subset) will lead to an increase in the smallest and largest stationary equilibrium aggregates.

The next comparative statics result predicts the effect of a change in agents’ level of patience. This result is not a corollary of the previous result because an increase in the discount factor is not covered by our notion of a positive shock.

**Theorem 5 (Discounting and Stationary Equilibrium)** Supposed Assumptions 1 and 2 hold for every agent $i$ and in addition assume that each $u_i(x_i, y_i, z_i, Q)$ is increasing in $x_i$ and that each $\Gamma_i$ is expansive in $x_i$ ($x_i \leq \tilde{x}_i \Rightarrow \Gamma_i(x_i, z_i, Q) \subseteq \Gamma_i(\tilde{x}_i, z_i, Q)$). Then an increase in the discount factor $\beta$ leads to an increase in the largest and smallest stationary equilibrium aggregates.

**Remark 4** If $u_i$ is decreasing in $x_i$ and $\Gamma_i$ is contractive in $x_i$ ($x_i \leq \tilde{x}_i \Rightarrow \Gamma_i(x_i, z_i, Q) \supseteq \Gamma_i(\tilde{x}_i, z_i, Q)$), the conclusion of the previous theorem changes to: Then an increase in the discount factor $\beta$ leads to a decrease in the largest and smallest stationary equilibrium aggregates. To see this, simply substitute $\tilde{y}_i = -y_i$ and $\tilde{x}_i = -x_i$ and follow the proof using the increasing value function $v^n(\tilde{x}_i, \beta)$ in order to conclude that $y_i = -G_i(-x_i, \beta)$ will be descending in $\beta$.

### 4.3 Individual Comparative Statics

The results provided so far hold without knowing how individual behavior response to the underlying changes in parameters (“positive shocks”). The remarkable feature of our results is that a lot can be known about how the aggregate behaves without this knowledge – in fact even when individual behavior can be very irregular. Nevertheless, under stronger assumptions we can also specify what happens to individual behavior as we do in the next theorem. Crucially, note that the strategy of proof is to go from the aggregate level to the individual level rather than the other way around as in standard approaches: Once we know that the market aggregate will increase, we can simply treat this as an exogenous variable to the individuals alongside the truly exogenous parameters. The individual comparative statics question then becomes a standard comparative statics problem where existing, very powerful results can be made to bear (Topkis (1978), Milgrom and Shannon (1994), and Quah (2007)).

**Theorem 6 (Individual Comparative Statics)** Suppose that an increase in $Q$ is a positive shock to player $i$ (i.e., $u_i$ and $\Gamma_i$ satisfy Definition 5 with $t = Q$). Then the increase in $Q$ of Theorems 4-5 will lead to an increase in $x_i$ in stationary equilibrium (the increase here is in the first-order stochastic dominance sense). If instead $Q$ constitutes a “negative shock” (if Definition
5 is satisfied with $t = -Q$; and player $i$ is not affected by any change in exogenous parameters, the increase in $Q$ of Theorem 4 will lead to a (first-order stochastic dominance) decrease in $x_i$ in stationary equilibrium.

Here we should mention that if $Q$ is a positive shock to all agents in the sense of theorem 6, we are in effect looking at a monotone/supermodular economy (and here the conclusions are known, though they aren’t stated overly clearly anywhere). If instead $Q$ is a negative shock for everyone, we are in a similar vein looking at a “submodular economy” (here no results exist). Notably, our main contribution in Theorem 4, requires neither (for all or even a single player).

5 Comparative Statics II: Changes in the Uncertainty Environment

In this section, we present our comparative statics results in response to changes in the distribution of the idiosyncratic shock processes. For the results in this section, the exogenous parameters $(a_i)_{i\in I}$ play no role at all, and we suppress them to simplify notation. The following additional assumption will be needed throughout:

**Assumption 3** $u_i(x_i, y_i, z_i, Q)$ exhibits increasing differences in $y_i$ and $z_i$, and $\Gamma_i(x_i, z_i, Q)$ is ascending in $z_i$.

When coupled with Assumption 2, Assumption 3 implies that the policy correspondence $G_i(x_i, z_i, Q, a_i)$ is ascending in $z_i$ (Hopenhayn and Prescott (1992)).

5.1 First-Order Stochastic Dominance Changes

We begin by looking at first-order stochastic dominance increases in the distribution of $z_{i,t}$ for all or a subset of the players. To get results in this situation we need an additional assumption involving once again Hopenhayn and Prescott (1992)’s notion of strict complementarities introduced at the beginning of Section 4.2. $\Gamma_i$ has strict complementarities in $(x_i, z_i)$ if (for any fixed value of $Q$), for all $x_i^2 \geq x_i^1$ and $z_i^2 \geq z_i^1$, $y \in \Gamma_i(x_i^1, z_i^2, Q)$ and $\tilde{y} \in \Gamma_i(x_i^2, z_i^1, Q)$ implies that $y \wedge \tilde{y} \in \Gamma_i(x_i^1, z_i^1, Q)$ and $y \vee \tilde{y} \in \Gamma_i(x_i^2, z_i^2, Q)$.

**Assumption 4** $\Gamma_i(x_i, z_i, Q, a_i)$ has strict complementarities in $(x_i, z_i)$.

Let stationary distributions of $z_i$, $\mu_{z_i}$ be ordered by first-order stochastic dominance. Together with Assumptions 2-3, the previous assumption ensures that the policy correspondence of player $i$, when parameterized by $\mu_{z_i}$, $G_i(x_{i,t}, z_{i,t}, \mu_{z_i})$ is ascending in $\mu_{z_i}$ (Hopenhayn and Prescott (1992)).
It is intuitively clear that when this is so, a first-order stochastic dominance increase in $\mu_i$ will lead to an increase in the affected player’s optimal strategy. Just as in our previous results, it is far from trivial that this will translate into an increase in the aggregate in equilibrium (see the discussion prior to Theorem 4 which applied word-by-word to the following result).

**Theorem 7 (Comparative Statics of a First-Order Stochastic Dominance Change)**

Under Assumptions 1, 2, 3, and 4, a first-order stochastic dominance increase in the stationary distribution of $z_{i,t}$ for all $i$ (or any subset hereof), will lead to an increase in the smallest and largest stationary equilibrium aggregates.

**Remark 5** It is straightforward to see that Theorem 6 carries over to this case to obtain individual comparative statics results once the change in the aggregate is determined.

**Remark 6** It should be noted that Assumption 4 may be quite restrictive in large economies because a first-order stochastic dominance increase in the stationary distribution of an agent’s idiosyncratic shocks may not lead to a first-order stochastic dominance increase in her strategy. This can be seen in the context of the Bewley-Aiyagari model introduced in Section 2.1 where:

$$\Gamma_i(x_i, z_i, Q) = \{y_i \in [-b_i, b_i] : y_i \leq r(Q)x_i + w(Q)z_i\}.$$  

Take $r(Q) = w(Q) = 1$ and $x_1^1 = 1$, $x_2^2 = 2$, $z_1^1 = 1$, and $z_2^2 = 3$. Then let $y = 4 \in \Gamma_i(x_1^1, z_1^1, Q) = [-b_i, 4]$ and $\tilde{y} = 3 \in \Gamma(x_2^2, z_1^1, Q) = [-b_i, 3]$. But it is clear then that $y \land \tilde{y} = 3 \notin \Gamma_i(x_1^1, z_1^1, Q) = [-b_i, 2]$, and so $\Gamma_i$ does not have strict complementarities in $(x_i, z_i)$. So in the Bewley-Aiyagari model any general results from first-order stochastic dominance changes in the noise environment are not possible. Nevertheless, interestingly, we will see that mean-preserving spreads lead to unambiguous changes in market aggregates without any need for strict complementarities in $(x_i, z_i)$.

### 5.2 Mean Preserving Spreads

We first introduce the convex order $\succeq_{cx}$ over the space of distributions such that $\mu_{z_i} \succeq_{cx} \mu'_{z_i}$ if and only if $\int f(\tau)\mu(\tau) \geq \int f(\tau)\mu'(\tau)$ for all convex functions $f$. It is well known that $\mu_{z_i} \succeq_{cx} \mu'_{z_i}$ if and only if $\mu_{z_i}$ is a mean-preserving spread of $\mu_{z_i}$. We will then look at changes in the stationary distribution of individual-level noise $z_i$. For example, in the Bewley-Aiyagari model where $z_i$ is the labor endowment/earnings, a mean-preserving spread intuitively means that consumers face increased uncertainty about their earnings with the mean staying the same.
We also need to introduce some more definitions and assumptions. Recall that a correspondence \( \Gamma : X \to 2^X \) has a \textit{convex graph} if for all \( x, \tilde{x} \in X \) and \( y \in \Gamma(x) \) and \( \tilde{y} \in \Gamma(\tilde{x}) \): 
\[
\lambda y + (1 - \lambda)\tilde{y} \in \Gamma(\lambda x + (1 - \lambda)\tilde{x}) \quad \text{for all } \lambda \in [0, 1].
\]

**Assumption 5**

1. \( X_i \subseteq \mathbb{R} \) for all \( i \).\(^{22}\)

2. \( \Gamma_i(\cdot, z_i, Q) : X_i \to 2^{X_i} \) and \( \Gamma_i(x_i, \cdot, Q) : Z_i \to 2^{X_i} \) have convex graphs and \( u_i(x_i, y_i, z_i, Q) \) is concave in \((x_i, y_i)\), strictly concave in \( y_i \), and is increasing in \( x_i \).

We also define:

**Definition 6** Let \( k \geq 0 \). A function \( f : X \to \mathbb{R}_+ \) is said to be \( k \)-convex [\( k \)-concave] if:

- When \( k \neq 1 \), the function \( \frac{1}{1-k} [f(x)]^{1-k} \) is convex [concave].
- When \( k = 1 \), the function \( \log f(x) \) is convex [concave] (i.e. \( f \) is log-convex [log-concave]).

A detailed treatment of the notions of \( k \)-convexity and \( k \)-concavity and further references can be found in Jensen (2011).

**Theorem 8** (The Comparative Statics Effect of Mean-Preserving Spreads) Suppose that Assumptions 1, 2, 3, and 5 hold for all agents, and in addition assume that each \( u_i \) is differentiable and satisfies the following upper boundary condition 
\[
\lim_{y_i^n \uparrow \sup \Gamma_i(x_i, z_i, Q)} D_{y_i} u_i(x_i, y_i^n, z_i, Q) = -\infty \quad \text{(which ensures that } \sup \Gamma_i(x_i, z_i, Q) \text{ will never be optimal given } (x_i, z_i, Q))
\]
Then a mean-preserving spread to the invariant distribution \( \mu_{z_i} \) of any subset of agents \( I' \subseteq I \) will lead to an increase in the largest and smallest stationary equilibrium aggregates if for each \( i \in I \), there exists a \( k_i \geq 0 \) such that \(-D_{y_i} u_i(x_i, y_i, z_i, Q)\) is \( k_i \)-concave in \((x_i, y_i)\) and \((y_i, z_i)\); and \( D_{y_i} u_i(x_i, y_i, z_i, Q)\) is \( k_i \)-convex in \((x_i, y_i)\) and \( k_i \)-convex in \((z_i, y_i)\).

Theorem 8 provides a fairly easy to apply result showing how changes in the individual-level noise affect market aggregates. Intuitively, mean preserving spreads increase individual level actions whenever the policy correspondence defined in (10) is convex in \( x_i \) (note that the policy correspondence will be single-valued/a function under Assumption 5, so this statement is unambiguous). The assumptions of Theorem 8 precisely ensure such convexity of policy functions (see Jensen (2011), and also Carroll and Kimball (1996) and Huggett (2004) for a detailed discussion).

\(^{22}\)This part of assumption is imposed for notational convenience and can be relaxed.
As in all the other results of this paper, it is non-trivial that the partial equilibrium/individual level outcome carries over to the market level - indeed, it is perfectly possible that some individuals because of the equilibrium increase in $Q$ end up actually lowering their strategies even though their initial response to the increase in uncertainty was to raise their strategies.

6 Applications

In this section we go back and verify in detail the comparative statics results already announced in Section 2. In both cases, we emphasize how the assumptions of the approach developed so far can be easily checked and in consequence, the theorems above lead to general comparative static results.

6.1 Comparative Statics in the Bewley-Aiagari Model

To exploit this paper’s comparative statics results, we must verify Assumptions 1-2. Assumption 1 is trivially satisfied under the general conditions (continuity, compactness) described in Section 2.1.

Assumption 2 requires that $u_i$ is supermodular in $(x_i, y_i)$ and that the graph of $\Gamma_i(\cdot, z_i, Q)$ is a sublattice of $X_i \times X_i$.\footnote{In addition, the choice set $X_i \subseteq \mathbb{R}$, must be a lattice. But this is is trivially satisfied whenever the choice set is one-dimensional, because any one-dimensional set is a lattice.} Beginning with supermodularity, this will hold if and only if the period utility function $v_i$ is concave. This equivalence is true in general, but it is particularly easy to see when $v_i$ is twice continuously differentiable since then $D^2v_i \leq 0$ (concavity) $\iff D^2u_i(x_i, y_i) \geq 0$ (supermodularity). Next turning to the sublattice property, as noted in Remark 3, $\Gamma_i(\cdot, z_i, Q)$ will be a sublattice of $X_i \times X_i$ if and only if $\Gamma_i(x_i, z_i, Q)$ is ascending in $x_i$ (this is true in general when $X_i$ is one-dimensional). Recall from Section 2.1 that

$$\Gamma_i(x_i, z_i, Q) = \{y_i \in [-\bar{b}_i, \bar{b}_i] : y_i \leq r(Q)x_i + w(Q)z_i\}.$$ 

This correspondence is ascending in $x_i$ if (for any fixed choice of $(z_i, Q)$) whenever $x_i^2 \geq x_i^1$, $y_i^1 \in \Gamma_i(x_i^1, z_i, Q)$, and $y_i^2 \in \Gamma_i(x_i^2, z_i, Q)$, we have $\max\{y_i^1, y_i^2\} \in \Gamma_i(x_i^1, z_i, Q)$ and $\min\{y_i^1, y_i^2\} \in \Gamma_i(x_i^2, z_i, Q)$. It is straightforward to see that this will indeed be the case, intuitively because $\Gamma_i$ is “increasing in $x_i$”.

We also note that $u_i$ is increasing in $x_i$ and that $\Gamma_i$ is expansive in $x_i$ (these additional properties are used in Theorem 5, where an expansive correspondence is also defined). Finally we recall from the discussion immediately prior to Definition 5, that a tightening of the borrowing limits (a decrease in the $b_i$’s) will be positive shocks.
Using our first set of comparative statics theorems in Section 4 (Theorems 4-5), we can then straightforwardly conclude (proof omitted):

**Proposition 1** Consider the generalized Bewley-Aiyagari model described in Section 2.1. The following then follow:

- An increase in the discount rate $\beta$ will lead to an increase in the smallest and largest capital-labor ratios in equilibrium, as well as an increase in the associated smallest and largest equilibrium output per capita.

- Any tightening of the borrowing limits (a decrease in $b_i$ for all or a subset of households) is a positive shock and consequently leads to an increase in the smallest and largest capital-labor ratios in equilibrium, as well as an increase in the associated smallest and largest equilibrium output per capita.

- Let $a_i$ parameterize the instant utility function $v_i = v_i(c_i, a_i)$ where $c_i$ denotes consumption at a point in time, and consider the effect of a decrease in marginal utility, i.e., assume that $D^2_{c_i a_i} v_i \leq 0$. Then an increase in $a_i$ (for any subset of the agents not of measure zero) will lead to an increase in the smallest and largest capital-labor ratios in equilibrium, as well as an increase in the associated smallest and largest equilibrium output per capita.

For example with $v_i(c_i, a_i) = \frac{1}{1-a_i} c_i^{1-a_i}$, we have $D^2_{c_i a_i} v_i(c_i, a_i) = -c_i^{-a_i} \log c_i \leq 0 \iff c_i \geq 1$. So a group of “sufficiently wealthy” consumers (whose consumption level never falls below unity) will increase their savings if they become more risk averse. But a group of “sufficiently poor” consumers (consumption levels always below unity), will lower their savings if they become more risk averse. With constant absolute risk aversion $v_i(c_i, a_i) = 1 - \exp(-a_i c_i)$ we get $D^2_{c_i a_i} v_i(c_i, a_i) = -a_i c_i \exp(-a_i c_i) \leq 0$ as long as $a_i, c_i \geq 0$, so an increase in the rate absolute risk aversion always leads to higher savings.

We can also use the results in Proposition 1 to briefly discuss why in general very little can be said about individual behavior even though quite strong results on aggregates are obtained. Consider, for example, an increase in $\beta$. At given $Q$, this will increase the savings (asset holdings) of all individuals and thus correspond to a positive shock in terms of our terminology. This will naturally tend to increase the aggregate capital-labor ratio. As the aggregate capital-labor ratio increases, the wage rate increases and the interest rate declines. But this might discourage savings by at least some of individuals. Even a small increase in $Q$ may have a significant impact on the savings of some individuals depending on income effects and substitution effects. Thus at the end a subset of individuals may end up reducing their savings and a subset may end up raising
savings (where for any specific agent, the outcome depends on the current level of assets and her underlying preferences). In fact, it is in general very difficult to say which individuals will reduce and which will increase their savings, because this will depend on the exact changes in the wage and interest rates. However, even though some individuals might reduce their savings and the extent of this is quite irregular, we know that in the aggregate savings and thus $Q$ must go up.

The economic intuition for this is that without the increase in $Q$, the changes in prices triggering the reduction in savings and some subset of agents would not have existed.

For our second set of comparative statics results in Section 5, we first need to verify Assumption 3. This requires that $u_i(x_i, y_i, z_i, Q)$ must exhibit increasing differences/be supermodular in $y_i$ and $z_i$, and that $\Gamma_i(x_i, z_i, Q)$ is ascending in $z_i$. But a quick look at the definitions of $u_i$ and $\Gamma_i$ will convince us that this is so by the exact same line of reasoning used a moment ago to conclude that $u_i$ is supermodular in $x_i$ and $y_i$ and that $\Gamma_i$ ascending in $x_i$ (this is simply because $x_i$ and $z_i$ enter in an entirely “symmetric” way in $u_i$ and $\Gamma_i$).

Remark 6 at the end of Section 5.1 noted that $\Gamma_i$ in the current model does not satisfy strict complementarities in $(x_i, z_i)$. Hence we cannot say anything about first-order stochastic dominance changes in the invariant distributions of the households’ stochastic processes.

Nevertheless, the effects of mean preserving spreads (in particular, a mean-preserving spread to $\mu_{z_i}$ for any subset of the agents) can be determined using the result of Section 5.2. Beginning with Assumption 5, it is straightforward to verify that $\Gamma_i$ has a convex graph as required. The concavity parts of Assumption 5 will all hold if we take $v_i$ to be strictly concave (note that this corresponds to assuming that households are risk averse). Next let us turn to the required $k$-concavity and $k$-convexity conditions of Theorem 8. Specifically, there must for each household $i$ exist an $k_i \geq 0$ such that $-D_{y_i}u_i(x_i, y_i, z_i, Q)$ is $k_i$-concave in $(x_i, y_i)$ as well as $(y_i, z_i)$ and $D_{x_i}u_i(x_i, y_i, z_i, Q)$ is $k_i$-convex in $(x_i, y_i)$ and convex in $(z_i, y_i)$. Now, because of the linear way $y_i$, $x_i$, and $z_i$ enter in the definition of $u_i$ in terms of $v_i$, it is easy to verify that all of these conditions will be satisfied simultaneously if and only if $Dv_i(c_i)$ is $k_i$-concave as well as $k_i$-convex. In other words, $\frac{1}{1-k_i}[Dv_i(c_i)]^{1-k_i}$ must be linear in $c_i$. Clearly, strict concavity in addition requires that $k_i > 0$. Differentiating twice, setting equal to zero, and rearranging this yields the condition:

$$\frac{D^3v_i(c_i)Dv_i(c_i)}{(D^2v_i(c_i))^2} = k_i > 0$$

This condition on a utility function is well known, and a function that satisfies it is said to belong to the HARA class (Carroll and Kimball (1996)). Most commonly used utility functions are in fact in the HARA class, including those that exhibit either constant absolute risk aversion (CARA) or constant relative risk aversion (CRRA). Note that, conveniently, such functions will
also satisfy the boundary condition of Theorem 8. So by picking \( v_i \) in the HARA class we in fact ensure that all of the conditions of Theorem 8 hold, and so we get:

**Proposition 2** Consider the generalized Bewley-Aiyagari model of Section 2.1, and assume that \( v_i \) belongs to the HARA class for all \( i \). Then a mean-preserving spread to (any subset of) the households’ noise environments will lead to an increase in the smallest and largest equilibrium capital-labor ratios and an increase in the associated smallest and largest equilibrium per capita outputs.

Proposition 2 shows that an observation made by Aiyagari (1994) (p. 671) with reference to his example depicted in the figures on p.668, is in fact true under very general conditions: an economy with idiosyncratic shocks will lead to higher savings and output per capita than a parallel economy without any uncertainty.\(^{24}\) Proposition 2 is also very closely related to a contribution by Huggett (Huggett (2004)), who shows that an individual agent’s accumulation of wealth will increase if she is subjected to higher earnings risk (in particular, this result is valid for preferences that are a subset of the HARA class, cf. Huggett (2004), p.776). Proposition 2 can be seen as “lifting” this individual-level result to the market/general equilibrium level. Note in this connection, that a crucial common component is that when utility belongs to the HARA class, the savings function will be convex, a result proved by Carroll and Kimball (1996) in the setting without borrowing constraints, and extended to the setting with borrowing constraint in Huggett (2004) and Jensen (2011).

### 6.2 Comparative Statics in the Hopenhayn Model

As explained in Section 2.2, Hopenhayn’s model of entry, exit, and firm dynamics can be cast as a large dynamic economy with the following aggregator \( H \) (cf. equation (??)):

\[
H((\tilde{x}_i(p))_{i\in I}) = D\left(\int_{i\in \mathcal{N}} \tilde{x}_i(p) \, di\right).
\]

Here \( \tilde{x}_i(p) \) is the strategy of a firm given the stationary price level \( p \). The only difference from the Bewley-Aiyagari model is that \( \tilde{x}_i(p) \) is now a random variable \( x^*(\cdot, p) \) defined on the probability space \( (Z, \mathcal{B}(Z), \mu_p) \), where \( \mu_p \) (the frequency distribution of the active firms’ productivities) in general will depend not only on \( p \) but on any exogenous parameters of the model. Therefore shocks will affect \( \tilde{x}_i(p) \) through two channels: directly through \( x^* \), and indirectly through the change in the distribution \( \mu_p \).

\(^{24}\)To see this, simply note that the movement from a deterministic model to one with uncertainty amounts to subjecting all agents’ labor endowments to mean preserving spreads.
It is easy to verify that Assumption 1 holds. Assumption 2 is also satisfied since for a given productivity level $z$, a firm will choose output simply so as to maximize $px - C(x, z, a) - c$ (here $a$ is an exogenous parameter affecting costs), and thus the payoff function only depends on $x$ and thus trivially satisfies the supermodularity assumption. Since there is no constraint other than $x \geq 0$ on this problem, the assumption that the graph of the constraint correspondence is a sublattice of $X_i \times X_i$ is also immediately satisfied. From this observation, it also follows that an increase in $a$ will be a positive shock if and only if $D^2_aC(x, z, a) \leq 0$. In other words, a positive shock is one that lowers the marginal cost (given $p$ and $z$). Let us also impose the natural restriction that $D_aC(x, z, a) \leq 0$ which implies that $V(z, p, a)$ is increasing in $a$.26

Next, note that, as outlined in Section 2.2, $\mu_p$ is determined from the exit cutoff $\tilde{z}_p$ and the measure of entrants $M$ as a solution to equation (4). The right-hand side of (4) is type I and type II monotone in $\mu_p$ as well as in $-\tilde{z}_p$ and $M$. Therefore Theorem 3 implies that an increase $M$ or a decrease in $\tilde{z}_p$ will lead to a (first-order stochastic dominance) increase in the distribution $\mu_p$.27

It follows that the aggregate in this case, $\int_z x^*(z, p)\mu_p(dz)$, will increase not only with positive shocks as defined above but also with other changes in parameters that lowers $\tilde{z}_p$ or raises $M$.28

Combining the previous observations, the following general predictions are now a consequence of theorem 4.

Proposition 3

1. A decrease in the fixed cost of operation $c$ or a (first-order) increase in the transition function $\Gamma$ increases aggregate output and lowers the equilibrium price.

2. A first-order stochastic increase in the entrants’ productivity distribution $\nu$ increases aggregate output and lowers the equilibrium price.

3. A positive shocks to the firms’ profit functions, i.e., an increase in $a$ with $D_aC \leq 0$ and $D^2_{xa}C \leq 0$, increases aggregate output and lowers the equilibrium price.

25 These observations also show that an interesting generalization of Hopenhayn’s model with learning by doing at the firm level—where current productivity depends on past production—is also a special case of our framework and will yield essentially the same comparative static results provided that the interaction between current output and past output satisfies supermodularity.

26 In this statement $\mu_p$ is ordered by first-order stochastic dominance. The right-hand side of (4), $F(\mu(\cdot), \tilde{z}_p, M) = \int_{z_i \geq \tilde{z}_p} \Gamma(z_i, \cdot)\mu(dz_i) + M\nu(\cdot)$, is single-valued, so type I and type II monotonicity coincide with monotonicity in the usual sense. Note that $\int_{z_i \geq \tilde{z}_p} \Gamma(z_i, \cdot)\mu(dz_i)$ is simply the adjoint of $\Gamma$ imputed at $\tilde{z}_p$. From this follows immediately that $F$ will be monotone in $\mu_p$ since $\Gamma$ is monotone (and it also easily follows that a decrease in $\tilde{z}_p$ will lead to a first-order stochastic increase in $F$). That $F$ is monotone in $M$ (as well as in $\nu$ ordered by first-order stochastic dominance) is straightforward to verify.

27 When $V(z, p, a)$ is increasing in $a$—which our assumption that $D_aC(x, z, a) \leq 0$ guarantees—an increase in $a$ will lead to an increase in $M$ (which can be directly seen from equation (3)), and thus to an increase in $\mu_p$.

28 The fact that this aggregate, $\int_z x^*(z, p)\mu_p(dz)$, increases when $\mu_p(z)$ undergoes a type I and/or type II increase is a consequence of Corollary 1.
It is useful to note that the results are truly equilibrium comparative statics. In fact, the effects on individual firms are uncertain, and may easily go in the opposite direction. Take a decline in the fixed costs of operation \( c \) to illustrate this for the first part of the proposition. Such a decline leaves the profit-maximizing choice of output for incumbents, \( x(p, z) \), unchanged for any given price and level of productivity. The conclusion in part 1 of Proposition 3 instead follows the effect of this cost reduction on the equilibrium distribution \( \mu_p \)—“state of the industry”. This is because as \( c \) declines, the value of a firm with any given productivity \( V(p, z) \) increases and the exit cutoff \( \bar{z}_p \) also decreases, making it less likely that any active firm will exit in any period. The increase in \( V(p, z) \) (for all \( z \)) also leads to greater entry, which together with the decline in \( \bar{z}_p \) leads to an increase in \( \mu_p \), raising aggregate output. But as aggregate output increases, the equilibrium price will fall which leads to counteracting effects on \( V(p, z) \) as well as \( \bar{z}_p \) (a decrease and an increase, respectively). The combined consequence for any firm with a given productivity level \( z \) is uncertain—for many types of firms the indirect effects may dominate, reducing their output, and some types of firms might choose to exit. Nevertheless, aggregate output necessarily increases and the equilibrium price necessarily declines. Similarly in part 2, the result is again driven by the impact of the shift in \( \nu \) on \( \mu_p \); the resulting decline in \( p \) is a counteracting effect, reducing firm-level output at given productivity level \( z \). In part 3, a positive shock directly raises \( x(p, z, a) \) for all \( p, z \) and also raises the value function \( V \), increasing \( \mu_p \), and thus aggregate output, and lowering the equilibrium price. Because the resulting decrease in \( p \) counteracts this effect, the overall impact on a firm of a given productivity level \( z \) is again uncertain. This discussion thus illustrates that the types of results contained in Proposition 3 would not have been possible by studying comparative statics at the individual firm level—indeed, similar with some of the results discussed in Proposition 1, there will generally be no regularity at the individual level.

7 Conclusion

There are relatively few known comparative static results on the structure of equilibria in dynamic economies. Many existing analytic results, such as those in endogenous growth models (overviewed in Acemoglu (2009)), are obtained using closed-form characterizations and rely heavily on functional forms. Many other works study the structure of such models using numerical analysis. In this paper, we developed a general and fairly easy-to-apply framework for robust comparative statics about the structure of stationary equilibria in such dynamic economies. Our results are “robust” in the sense defined by Milgrom and Roberts (1994) in that they do not rely on parametric assumptions but on qualitative economic properties, such as utility functions.
exhibiting increasing differences in choice variables and certain parameters. Nevertheless, and importantly from the viewpoint of placing the contribution within the broader literature, none of our main results exploit standard supermodularity or monotonicity results—and in fact, our key technical result, which underlies all of our results, is introduced to enable us to work with spaces that are not lattices.

Some of the well-known models that are special cases of our framework are models of saving and capital accumulation with incomplete markets along the lines of work by Bewley, Aiyagari, and Huggett, and models of industry equilibrium along the lines of work by Hopenhayn. In all cases, our results enable us to establish much stronger and more general results than those available in the literature (to the best of our knowledge). They also lead to a new set of comparative static results in response to first-order and second-order stochastic dominance shifts in distributions representing uncertainty in these models. All of the major comparative static results provided in the paper are truly about the structure of equilibrium—not about individual behavior. This is highlighted by the fact that in most cases, while robust and general results can be obtained about how market outcomes behave, little can be said about individual behavior, which is in fact often quite irregular.

We believe that our framework and methods are useful both because they clarify the underlying economic forces, for example in demonstrating that robust comparative statics applies to aggregate market variables and not to individual decisions, and because they can be applied readily in a range of problems.

8 Appendices

8.1 Appendix I: Proofs of Results from the Text

In this Appendix, we present the proofs of the main results from the text. Some of these proofs rely on technical results presented in Appendixes II and III.

8.1.1 Proofs from Section

Proof of Theorem 1. Only a brief sketch will be provided. For agent $i$, let $\mathcal{X}_i$ denote the set of strategies (these are infinite sequences of random variables as described above), and let $\gamma_i(Q) \subseteq \mathcal{X}_i$ denote that set of optimal strategies for agent $i$ given the sequence of aggregates $Q \in \prod_{t=0}^{\infty} Q$. $\prod_{t=0}^{\infty} Q$ with the supremum norm $\|Q\| = \sup_t |Q_t|$, is a compact and convex topological space. $\mathcal{X}_i$ is equipped with the topology of pointwise convergence where each coordinate converges if and only if the random variable converges in the weak $*$-topology. Under Assumption 1, $\gamma_i : \prod_{t=0}^{\infty} Q \to 2^{\mathcal{X}_i}$ will be non-empty valued and upper hemi-continuous. Let $\mathcal{H}(Q) = \{H((x_i)_{i \in I}) :$
\[ x_i \in \gamma_i(Q) \text{ for } i \in I \}. \] Since \( H \) is continuous and convexifying, \( H \) will be upper hemi-continuous and convex valued. A fixed point \( Q^* \in H(Q^*) \) exists by the Kakutani-Glicksberg-Fan Theorem. It is easy to see that such a \( Q^* \) corresponds to an equilibrium with \( x_i^* \in \gamma_i(Q^*) \) for each agent \( i \).

8.1.2 Proofs from Subsection 4.1

The proof of Theorem 3 relies on a generalization of the following result from Smithson (1971).

**Theorem 9 (Smithson (1971))** Let \( X \) be a chain-complete partially ordered set, and \( F : X \rightarrow 2^X \) a type I monotone correspondence. Assume as follows: For any chain \( C \) in \( X \), and any monotone selection from the restriction of \( F \) to \( C \), \( f : C \rightarrow X \) (if one exists!); there exists \( y_0 \in F(\sup C) \) such that \( f(x) \leq y_0 \) for all \( x \in C \). Then, if there exists a point \( e \in X \) and a point \( y \in F(e) \) such that \( e \preceq y \), \( F \) has a fixed point.

The generalization, which to the best of our knowledge is new, is presented and proved next.

**Theorem 10** In Theorem 9, the conclusion may be strengthened to: \( F \) has a fixed point \( x^* \) with \( x^* \succeq e \).

**Proof.** Let \( F : X \rightarrow 2^X \) and \( e \in X \) be as described, and set \( \hat{X} \equiv \{ x \in X : x \succeq e \} \). Note that since \( X \) is chain-complete, so is \( \hat{X} \). Then define a correspondence on \( \hat{X} \) by \( \hat{F}(x) \equiv F(x) \cap \{ z \in X : z \succeq e \} \). We begin by showing that \( \hat{F} \) has non-empty values. So pick any \( x \in \hat{X} \). By type I monotonicity, there exists \( y' \in F(x) \) with \( y' \succeq y \preceq e \) where \( y \) is the element in \( F(e) \) with \( e \preceq y \) guaranteed to exist by assumption. But then \( y' \in \hat{F}(x) \). Next, \( \hat{F} : \hat{X} \rightarrow 2^{\hat{X}} \) is type I monotone, for if \( x_1 \preceq x_2 \) and \( y_1 \in \hat{F}(x_1) \subseteq F(x_1) \), there will exist \( y_2 \in F(x_2) \) such that \( y_1 \preceq y_2 \); and since \( e \preceq y_1 \preceq y_2 \), \( y_2 \in \hat{F}(x_2) \) also. That \( \hat{F} \) satisfies the condition on the supremum of chains in Theorem 9 is trivial to show and we omit the proof. Now all we have to do is apply Smithson (1971)’s Theorem in order to conclude that \( \hat{F} \) has a fixed point \( x^* \in \hat{X} \). But it is clear that any fixed point for \( \hat{F} \) is also a fixed point for \( F \), and since by construction \( x^* \succeq e \), this completes the proof.

Both of the previous results have parallel statements for type II monotone correspondences. In particular (see Smithson (1971), Remark p. 306), the conclusion of Theorem 9 (existence of a fixed point) remains valid for type II monotone correspondences if the hypothesis are altered as follows: (i) \( X \) is assumed to be lower chain-complete rather than chain-complete (a partially ordered set is lower chain complete if each non-empty chain has an infimum). (ii) The condition on monotone selections on chains is altered to: For any chain \( C \) in \( X \), and any monotone selection
from the restriction of $F$ to $C$, $f : C \to X$ (if any); there exists $y_0 \in F(\inf C)$ such that $f(x) \geq y_0$ for all $x \in C$. (iii) Instead of elements $e \in X$ and $y \in F(e)$ with $e \preceq y$; there must exist $e \in X$ and $y \in F(e)$ with $e \succeq y$. As for our Theorem 10, the conclusion will now be the existence of a fixed point $x^*$ with $x^* \preceq e$.

**Proof of Theorem 3.** We prove only the type I monotone case (the type II monotone case is similar). Compactness of $X$ together with the fact that the order $\succeq$ is assumed to be closed, ensures the chain-completeness as well as lower chain-completeness of $(X, \succeq)$.

The condition in Theorem 9 on the supremum (and infimum in the type II case) of chains is satisfied because $F$ is upper hemi-continuous. Indeed, let $C$ be a chain with supremum $\sup C \in X$, and let $f : C \to X$ be a monotone selection from $F : C \to 2^X$. There will exist an increasing sequence $(c_n)_{n=1}^{\infty}$, $c_{n+1} \succeq c_n$, from $C$ with $\lim_{n \to \infty} c_n = \sup C$. It follows then from upper hemi-continuity of $F$ that $f(\sup C) = \lim_{n \to \infty} f(c_n) \in F(\sup C)$. In addition, since $(f(c_n))_n$ is increasing with supremum $f(\sup C) = \lim_{n \to \infty} f(c_n)$, $f(\sup C) \succeq f(x)$ for all $x \in C$. This proves the claim.

Pick $t_1 \leq t_2$ and a fixed point $x_1 \in \Lambda(t_1)$. We must show that there will exist an $x_2 \in \Lambda(t_2)$ with $x_1 \preceq x_2$. To this end we apply Theorem 10 to the correspondence $F(\cdot, t_2)$. The only thing we need to verify is that there exists $e \in X$ and $y \in F(e, t_2)$ with $y \succeq e$. But taking $e = x_1$, it is clear that $e \in F(e, t_1)$, and since $F$ is type I monotone in $t$, there will for our $t_2 \geq t_1$ exist $y \in F(e, t_2)$ with $y \succeq e$. This is exactly what we needed. We conclude that $F(\cdot, t_2)$ has a fixed point “above” $e = x_1$, i.e., there exists $x_2 \in \Lambda(t_2)$ with $x_2 \succeq x_1$. □

**Proof of Corollary 1.** We prove only that $\overline{h}(t)$ is increasing (the other case is similar). $\overline{h}(t)$ is well-defined because $H$ is continuous and $\Lambda(t)$ is compact (the fixed point set of an upper hemi-continuous correspondence on a compact set is always compact). Pick $t_1 \leq t_2$, and let $x_1 \in \Lambda(t_1)$ be an element such that $\overline{h}(t) = H(x_1)$. Since $\Lambda(t)$ is type I monotone, there will exist $x_2 \in \Lambda(t_2)$ such that $x_2 \succeq x_1$. Since $H$ is monotone, $\overline{h}(t_2) = \sup_{x \in \Lambda(t_2)} H(x) \succeq H(x_2) \succeq H(x_1) = \overline{h}(t_1)$. □

**8.1.3 Proofs from Subsection 4.2**

**Proof of Theorem 4.** Fix $Q \in Q$. Under Assumption 2, the policy correspondence of each player $G_i : X_i \times Z_i \times \{Q\} \times A_i \to 2^{X_i}$ will have a least and a greatest selection and both of these will be increasing in $x_i$. For given $Q$ and $a_i$, let $T_{Q,a_i}^* : \mathcal{P}(X_i) \to 2^{\mathcal{P}(X_i)}$ denote the adjoint Markov correspondence induced by $G_i$. By Theorem 11, $T_{Q,a_i}^*$ will be type I and type II monotone when $\mathcal{P}(X_i)$ is equipped with the first-order stochastic dominance order $\succeq_{st}$.  

\[29\text{A partially ordered set where all chains have an infimum as well as a supremum is usually simply said to be complete (e.g., Ward (1954), p.148). In the present setting where }X\text{ is topological and the order }\succeq \text{ is closed, the claim that compactness implies completeness follows from Theorem 3 in Ward (1954) because any closed chain will be compact (any closed subset of a compact set is compact).}\]
Since \((\mathcal{P}(X_i), \succeq_a)\) has an infimum (namely the degenerate distribution placing probability 1 on \(\inf X_i\)), this implies that the invariant distribution correspondence \(F_i: Q \times A_i \to 2^{\mathcal{P}(X_i)}\), given by \(F_i(Q, a_i) = \{\mu \in \mathcal{P}(X_i) : \mu \in T_{Q,a_i}^*\}\) is non-empty valued and upper hemicontinuous (Theorem 12). Now the results from Section ?? come into play. Since, again by Theorem 11, \(T_{Q,a_i}^*\) is also type I and type II monotone in \(a_i\), we can use Theorem 3 to conclude that the invariant distribution correspondence \(F_i\) will be type I and type II monotone in \(a_i\). This is true for every \(i \in I\) hence the joint correspondence: \(F = (F_i)_{i \in I} : Q \times (\prod_{i \in I} A_i) \to 2^{\prod_{i \in I} \mathcal{P}(X_i)}\) is type I and type II monotone in \(a = (a_i)_{i \in I}\). Now consider:

\[
\hat{H}(Q, a) = \{H(x) \in \mathbb{R} : x \in F(Q, a) \text{ for all } i\}
\]

It is clear from the definition of a stationary equilibrium, that \(Q^*\) is a stationary equilibrium aggregate given \(a \in A\) if and only if \(Q^* \in \hat{H}(Q^*, a)\). If each \(G_i\) is convex valued, \(F\) and therefore \(\hat{H}\) will have convex values (this observation is important if the theorem is applied to a situation with only a finite number of players). Under this paper’s general assumptions, \(F_i\) will in general not have convex values, but \(\hat{H}\) will be convex valued due to the convexifying property of the aggregator \(H\). Since \(H\) is continuous and each \(F_i(Q, a_i)\) is upper hemicontinuous, \(\hat{H}\) will in addition be upper hemicontinuous (in particular, it has a least and a greatest selection). Since \(F\) is type I and type II monotone, and \(H\) is increasing, we can next use Theorem 1 to conclude that \(\hat{H}\)’s least and greatest selections will be increasing. Finally, let \(Q_{\min} = H(\delta_{\inf X_i})_{i \in I}\) and \(Q_{\min} = H(\delta_{\sup X_i})_{i \in I}\) where \(\delta_{x_i}\) denotes the degenerate measure on \(X_i\) with its mass at \(x_i\). It is then clear that \(Q \geq Q_{\min}\) for all \(Q \in \hat{H}(Q_{\min})\) and \(Q \leq Q_{\max}\) for all \(Q \in \hat{H}(Q_{\max})\).

That the least and greatest solutions to the fixed-point problem \(Q^* \in \hat{H}(Q^*, a)\) are increasing in \(a\) now follows from a general result: If \(\hat{H}: [Q_{\min}, Q_{\max}] \to [Q_{\min}, Q_{\max}]\) is upper hemicontinuous, convex valued, and has least and greatest selections that are increasing in \(a\), then the least and greatest solutions will be increasing in \(a\) (this follows from Milgrom and Roberts (1994), Corollary 2 once it has been shown that the conditions of that result will be satisfied in the case at hand (see Acemoglu and Jensen (2009), the proof of Lemma 2, for details). ■

**Proof of Theorem 5.** The value function of agent \(i\) will, given a stationary sequence for the aggregate \(Q_t = Q\) all \(t\), and the stationary distribution for \(z_{i,t}, z_i \sim \mu_{z_i} \in \mathcal{P}(Z_i)\) all \(t\), equal the pointwise limit of the sequence \((v^n_i)_{n=0}^{\infty}\) determined by:

\[
v_i^{n+1}(x_i, z_i, \beta) = \sup_{y_t \in G_i(x_i, z_i)} \left[u_i(x_i, y_i, z_i) + \beta \int v^n_i(y_i, z_i', \beta) \mu_{z_i}(dz'_i)\right]
\]

where \(v^0\) may be picked arbitrarily and we have suppressed \(Q\)’s entry to simplify notation. Pick \(v^0(x_i, z_i, \beta)\) that is increasing and supermodular in \(x_i\) and exhibits increasing differences in \(x_i\) and
β. Since integration preserves supermodularity and increasing differences, \( \int v^0(y_i, z'_i, \beta) \mu_{z_i}(dz'_i) \) will be supermodular in \( y_i \) and exhibit increasing differences in \( y_i \) and \( \beta \). It immediately follows from Topkis’ Theorem on preservation of supermodularity (Topkis (1998), Theorem 2.7.6), that \( v^1 \) will be supermodular in \( x_i \). By recursion then, \( v^2, v^3, \ldots \) are all supermodular in \( x_i \) and so is consequently the pointwise limit \( v^* \) (Topkis (1998), Lemma 2.6.1). It is trivial to show that when \( u_i \) is increasing in \( x_i \) and \( \Gamma_i \) is expansive in \( x_i \), the conclusion of the Theorem now follows from the exact same argument used to prove Theorem 4 (thinking of an increase in \( \beta \) as a “positive shock” or more precisely as leading to an increase in \( G_i(x_i, z_i, \beta) \)).

Proof of Theorem 6. The conclusions follow from the first part of the proof of Theorem 4 since \( Q \) can now be treated as an exogenous variable (alongside \( t \)) from the point of view of any individual agent.

8.1.4 Proofs from Section 5

In this part of Appendix I, we provide a proof of Theorem 8. We begin by noting that under Assumption 5, the policy correspondence of (10) will be single-valued, i.e., \( G_i(x_i, z_i, Q) = \{g_i(x_i, z_i, Q)\} \) where \( g_i \) is the (unique) policy function. For a given stationary market aggregate \( Q \in Q \), an agent’s optimal strategy is therefore described by the following stochastic difference equation:

\[
x_{i,t+1} = g_i(x_{i,t}, z_{i,t}, Q, \mu_{z_i})
\]

(13)

Note that here we have made \( g_i \)’s dependence on the distribution of \( z_{i,t} \) explicit. We already know that \( g_i \) will be increasing in \( x_i \) and \( z_i \) (Assumptions 2-3). By Theorem 8 of Jensen (2011), \( g_i \) will in addition be convex in \( x_i \) as well as in \( z_i \) under the conditions of the theorem. We now turn to proving that \( g_i \) will be \( \succeq_{cx} \)-increasing in \( \mu_{z_i} \) (precisely, this means that \( g_i(x_{i,t}, z_{i,t}, Q, \tilde{\mu}_{z_i}) \geq g_i(x_{i,t}, z_{i,t}, Q, \mu_{z_i}) \) whenever \( \tilde{\mu}_{z_i} \succeq_{cx} \mu_{z_i} \)). From Jensen (2011) (corollary in the proof of Theorem 30 Let \( f(y, \beta) \) exhibit increasing differences and be increasing in \( y \). Then \( \beta f(\tilde{y}, \beta) - \beta f(y, \beta) \) is clearly increasing in \( \beta \) for \( \tilde{y} \geq y \), showing that \( \beta f(y, \beta) \) exhibits increasing differences.

33
8 applied with \( k = 0 \), \( D_x v_i(x_i, z_i, Q) \) will (in the sense of agreeing with a function with these properties almost everywhere) be convex in \( z_i \) because \( D_x u_i(x_i, y_i, z_i, Q) \) is non-decreasing in \( y_i \) and convex in \((z_i, y_i)\). This is precisely one of the conditions of the following lemma (the other is supermodularity, already used). The lemma is stated in some generality because it is of independent interest (note that \( Q \) is suppressed in the lemma’s statement).

**Lemma 1** Assume that \( u_i(x_i, y_i, z_i) \) is supermodular in \((x_i, y_i)\) and denote the value function by \( v_i(x_i, z_i, \mu_{z_i}) \) where \( \mu_{z_i} \) is the stationary distribution of \( z_i \). Let \( x_i \) be ordered by the usual Euclidean order and \( \mu_{z_i} \) be ordered by \( \succeq_{ce} \). Then the value function exhibits increasing differences in \( x_i \) and \( \mu_{z_i} \) if for all \( \tilde{x}_i \geq x_i \) the following function is convex in \( z_i \) (for all fixed \( \mu_{z_i} \)):

\[
v_i(\tilde{x}_i, z_i, \mu_{z_i}) - v_i(x_i, z_i, \mu_{z_i})
\]

When the value function \( v_i(x_i, z_i, \mu_{z_i}) \) exhibits increasing differences in \( x_i \) and \( \mu_{z_i} \) it in turn follows that \( \int v_i(y_i, z'_i, \mu_{z_i}) \mu_{z_i}(dz'_i) \) exhibits increasing differences in \( y_i \) and \( \mu_{z_i} \) and so if \( v_i \) is supermodular in \( y_i \), the policy function \( g_i(x_i, z_i, \mu_i) \) will be increasing in \( \mu_i \).

**Proof.** Let \( v^n_i \) denote the \( n \)'th iterate of the value function and consider the \( n + 1 \)'th iterate

\[
v^{n+1}_i(x, z, \mu_{z_i}) = \sup_{y \in \Gamma_i(x, z)} \{u_i(x, y, z) + \beta \int v^n_i(y, z', \mu_{z_i}) \mu_{z_i}(dz')\}.
\]

Assume by induction that \( v^n_i \) exhibits increasing differences in \((y, \mu_{z_i})\) and that the hypothesis of the theorem holds for \( v^n_i \). When \( \tilde{y} \geq y \) and \( \mu_{z_i} \succeq_{ce} \mu'_{z_i} \) we then have

\[
\int v^n_i(\tilde{y}, z', \mu_{z_i}) - v^n_i(y, z', \mu_{z_i}) \mu_{z_i}(dz') \geq \int v^n_i(\tilde{y}, z', \mu_{z_i}) - v^n_i(y, z', \mu'_{z_i}) \mu'_{z_i}(dz') \geq \int v^n_i(\tilde{y}, z', \mu'_{z_i}) - v^n_i(y, z', \mu'_{z_i}) \mu'_{z_i}(dz').
\]

Here the first inequality follows from the definition of the convex order, and the second inequality follows from increasing differences of \( v^n_i \) in \((y, \mu_{z_i})\). Note that this evaluation implies the second conclusion of the lemma once the first has been established. Since \( u_i(x, y, z) + \beta \int v^n_i(y, z', \mu_{z_i}) \mu_{z_i}(dz') \) is supermodular in \((x, y)\) by assumption and trivially exhibits increasing differences in \((x, \mu_{z_i})\) it follows from the preservation of increasing differences under maximization that \( v^{n+1}(x, z, \mu_{z_i}) \) exhibits increasing differences in \((x, \mu_{z_i})\). The first conclusion of the lemma now follows from a standard argument (increasing differences is a property that is pointwise closed and the value function is the pointwise limit of the sequence \( v^n, n = 0, 1, 2, \ldots \)). ■

**Proof of Theorem 8:**

We begin with some notation. For a set \( Z \), let \( \mathcal{P}(Z) \) denote the set of probability distributions on \( Z \) with the Borel algebra. A distribution \( \lambda \in \mathcal{P}(Z) \) is larger than another probability distribution \( \tilde{\lambda} \in \mathcal{P}(Z) \) in the monotone convex order (written \( \lambda \succeq_{ccx} \tilde{\lambda} \)) if \( \int_Z f(\tau) \lambda(d\tau) \geq \int_Z f(\tau) \tilde{\lambda}(d\tau) \) for all convex and increasing functions \( f : Z \to \mathbb{R} \) for which the integrals exist (see Huggett (2004) and Shaked and Shanthikumar (2007), Chapter 4.A). The stochastic difference equation
(13) gives rise to a transition function \( P_{Q,\mu_{z_i}} \) in the usual way (here \( x_i \in X_i \) and \( A_i \) is a Borel subset of \( X_i \)):

\[
P_{Q,\mu_{z_i}}(x_i, A) \equiv \mu_{z_i}(\{ z_i \in Z_i : g_i(x_i, z_i, Q, \mu_{z_i}) \in A \}) \quad (14)
\]

This in turn determines the adjoint Markov operator:

\[
T^*_{Q,\mu_{z_i}} \mu_{x_i} = \int P_{Q,\mu_{z_i}}(x_i, \cdot) \mu_{x_i}(dx_i) \quad (15)
\]

\( \mu_{x_i}^* \) is an invariant distribution for (13) if and only if it is a fixed point for \( T^*_{Q,\mu_{z_i}} \), i.e., \( \mu_{x_i}^* = T^*_{Q,\mu_{z_i}} \mu_{x_i} \). We are first going to use that \( g_i \) is convex and increasing in \( x_i \) to show that \( T^*_{Q,\mu_{z_i}} \) will be a \( \succeq_{\text{cxt}} \)-monotone operator, i.e., we are going to show that \( \bar{\mu}_{x_i} \succeq_{\text{cxt}} \mu_{x_i} \Rightarrow T^*_{Q,\mu_{z_i}} \bar{\mu}_{x_i} \succeq_{\text{cxt}} T^*_{Q,\mu_{z_i}} \mu_{x_i} \).

The statement that \( T^*_{Q,\mu_{z_i}} \bar{\mu}_{x_i} \succeq_{\text{cxt}} T^*_{Q,\mu_{z_i}} \mu_{x_i} \) by definition means that for all convex and increasing functions \( f : X_i \to \mathbb{R} \):

\[
\int f(\tau) T^*_{Q,\mu_{z_i}} \bar{\mu}_{x_i}(d\tau) \geq \int f(\tau) T^*_{Q,\mu_{z_i}} \mu_{x_i}(d\tau)
\]

But since this is equivalent to,

\[
\int_{Z_i} \left[ \int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i})) \bar{\mu}_{x_i}(dx_i) \right] \mu_{z_i}(dz_i) \geq \int_{Z_i} \left[ \int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i})) \mu_{x_i}(dx_i) \right] \mu_{z_i}(dz_i)
\]

we immediately see that this inequality will hold whenever \( \bar{\mu}_{x_i} \succeq_{\text{cxt}} \mu_{x_i} \) (the composition of two convex and increasing functions is convex and increasing). This proves that \( T^*_{Q,\mu_{z_i}} \) is a \( \succeq_{\text{cxt}} \)-monotone operator.

Our next objective is to prove that \( \bar{\mu}_{z_i} \succeq_{\text{cxt}} \mu_{z_i} \Rightarrow T^*_{Q,\bar{\mu}_{z_i}} \mu_{x_i} \succeq_{\text{cxt}} T^*_{Q,\mu_{z_i}} \mu_{x_i} \) for all \( \mu_{x_i} \in \mathcal{P}(X_i) \).

As above, we can rewrite the statement that \( T^*_{Q,\bar{\mu}_{z_i}} \mu_{x_i} \succeq_{\text{cxt}} T^*_{Q,\mu_{z_i}} \mu_{x_i} \):

\[
\int_{Z_i} \left[ \int_{X_i} f(g_i(x_i, z_i, Q, \bar{\mu}_{z_i})) \mu_{x_i}(dx_i) \right] \bar{\mu}_{z_i}(dz_i) \geq \int_{Z_i} \left[ \int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i})) \mu_{x_i}(dx_i) \right] \mu_{z_i}(dz_i)
\]

\( (16) \)

Since \( f \) is increasing and \( g_i \) is \( \succeq_{\text{cxt}} \)-increasing in \( \mu_{z_i} \), it is obvious that for all \( z_i \in Z_i \):

\[
\int_{X_i} f(g_i(x_i, z_i, Q, \bar{\mu}_{z_i})) \mu_{x_i}(dx_i) \geq \int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i})) \mu_{x_i}(dx_i)
\]

Hence:

\[
\int_{Z_i} \left[ \int_{X_i} f(g_i(x_i, z_i, Q, \bar{\mu}_{z_i})) \mu_{x_i}(dx_i) \right] \bar{\mu}_{z_i}(dz_i) \geq \int_{Z_i} \left[ \int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i})) \mu_{x_i}(dx_i) \right] \bar{\mu}_{z_i}(dz_i)
\]

\( (17) \)

But we also have:

\[
\int_{Z_i} \left[ \int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i})) \mu_{x_i}(dx_i) \right] \bar{\mu}_{z_i}(dz_i) \geq \int_{Z_i} \left[ \int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i})) \mu_{x_i}(dx_i) \right] \mu_{z_i}(dz_i)
\]

\( (18) \)

\( ^{31} \)To verify (18), reverse the order of integration and use the convexity of \( f(g_i(\cdot, \cdot, Q, \bar{\mu}_{z_i})) \) and the definition of \( \succeq_{\text{cxt}} \).

35
Combining (17) and (18) we get (16) under the condition that $\tilde{\mu}_z \succeq_{cx} \mu_z$. This is what we wanted to prove.

We are now ready to use Theorem 3 to conclude that $F_i(Q, \mu_z) \equiv \{ \mu_x \in P(X_i) : \mu_x = T_{Q, \mu_z} \mu_x \}$ will be type I and type II monotone in $\mu_z$ when $P(Z_i)$ is equipped with the order $\succeq_{cx}$ and $P(X_i)$ is equipped with $\succeq_{cxi}$. Note that in the language of Theorem 3, $F$ equals $\{ T_{\ast}^\ast Q, \mu_z \}$ and $t$ corresponds to $\mu_z$.

The rest of the proof proceeds exactly as the proof of Theorem 4 with $(\mu_z)_{i \in I}$ replacing the exogenous variables $(a_i)_{i \in I}$ in that proof. To be a bit more specific, we let $F(Q, \mu_z) = \{(F_i(Q, \mu_z))_{i \in I} \}$ where $\mu_z = (\mu_z)_i \in \mathcal{I}$ and consider:

$$\hat{H}(Q, a) = \{ h(x) \in \mathbb{R} : x \in F(Q, \mu_z) \text{ for all } i \}$$

The rest of the proof then follows the proof of Theorem 4 line-by-line except that, as mentioned, $\mu_z$ replaces $a$. We are thus able to conclude that a mean-preserving spread to (any subset of) the agents leads to an increase in the smallest and the largest equilibrium aggregates.

### 8.2 Appendix II: Dynamic Programming with Transition Correspondences

Consider a standard recursive stochastic programming problem with functional equation:

$$v(x, z) = \sup_{y \in \Gamma(x, z)} [u(y, x, z) + \beta \int v(y, z') \mu_z(dz')]$$

As is well known, (19) has a unique solution $v^* : X \times Z \to \mathbb{R}$ (and this will be a continuous function) when $u : X^2 \times Z \to \mathbb{R}$ and $\Gamma : X \times Z \to 2^X$ are continuous, $X$ and $Z$ are compact, and $\beta \in (0, 1)$ (Stokey and Lucas (1989)). From $v^*$, the policy correspondence $G : X \times Z \to 2^X$ is then defined by,

$$G(x, z) = \arg \sup_{y \in \Gamma(x, z)} u(y, x, z) + \beta \int v^*(y, z') \mu_z(dz')$$

Clearly, $G$ will be upper semi-continuous under the above assumptions. A policy function is a measurable selection from $G$, i.e., a measurable function $g : X \times Z \to X$ such that $g(x, z) \in G(x, z)$ in $X \times Z$. Throughout it is understood that $X \times Z$ is equipped with the product $\sigma$-algebra, $\mathcal{B}(X) \otimes \mathcal{B}(Z)$. Recall that a correspondence such as $G$ is (upper) measurable if the inverse image of every open set is measurable, that is if $G^{-1}(O) \equiv \{(x, z) \in X \times Z : G(x, z) \cap O \neq \emptyset \} \in \mathcal{B}(X) \otimes \mathcal{B}(Z)$, whenever $O \subseteq X$ is open. An upper hemi-continuous correspondence is measurable (Aubin and Aubin and

---

$\succeq_{cxi}$ is a closed order on $P(X_i)$.
Frankowska (1990), Proposition 8.2.1). Since a measurable correspondence has a measurable selection (Aubin and Frankowska (1990), Theorem 8.1.3.), any upper hemi-continuous policy correspondence admits a policy function $g$. Let $\mathcal{G}$ denote the set of measurable selections from $G$, which was just shown to be non-empty.

Given a policy function $g \in \mathcal{G}$, an $x \in X$, and a measurable set $A \in \mathcal{B}(X)$ let:

$$
P_g(x, A) \equiv \mu_z(\{z \in Z : g(x, z) \in A\}) \quad \left(= \int_Z \chi_A(g(x, z))\mu_z(dz)\right)$$

(21)

For fixed $x \in X$, $P_g(x, \cdot)$ is a measure and for fixed $A \in \mathcal{B}(X)$, $P_g(\cdot, A)$ is measurable (the last statement is a consequence of Fubini’s Theorem). So $P_g$ is a transition function.

The family of policy correspondences $\mathcal{G}$ then gives rise to the transition correspondence:

$$
P(x, \cdot) = \{P_g(x, \cdot) : g \in \mathcal{G}\}$$

Intuitively, given a state $x_t$ at date $t$, there is a set of possible probability measures $P(x, \cdot)$ each of which may describe the probability of being in a set $A \in \mathcal{B}(X)$ at date $t + 1$.

**Lemma 2 (The Transition Correspondence is Upper Hemi-Continuous)** Consider a sequence $(x_n)_{n=0}^{\infty}$ in $X$ that converges to a limit point $x \in X$. Let $P_{g_n}(x_n, \cdot) \in P(x_n, \cdot)$ be an associated sequence of transition functions from the transition correspondence $P$. Then for any weakly convergent subsequence $P_{g_{n_m}}(x_{n_m}, \cdot)$ there exists a $P_g(x, \cdot) \in P(x, \cdot)$ such that $P_{g_{n_m}}(x_{n_m}, \cdot) \rightarrow_w P_g(x, \cdot)$.

**Proof.** We loose no generality by assuming that the original sequence actually converges, $P_{g_n}(x_n, \cdot) \rightarrow_w \mu$, where $\mu$ is a probability measure on $(X, \mathcal{B}(X))$. Precisely, this means that for all $f \in \mathcal{C}(X)$ (the set of continuous real-valued functions on $X$):

$$
\lim_{n \rightarrow \infty} \int f(z)P_{g_n}(x_n, dz) = \int f(z)\mu_z(dz)
$$

We must show that that this equality holds with $\mu_z(\cdot) = P_g(x, \cdot)$ for some $g \in \mathcal{G}$. Fix $z \in Z$ and consider the sequence $g_n(x_n, z)$, $n = 0, 1, 2, \ldots$. By the upper hemi-continuity of $G$, $\lim_{n \rightarrow \infty} g_n(x_n, z) \in G(x, z)$ (passing, if necessary to a subsequence which we index here again by $n$ to simplify notation). Then let $g(x, z) = \lim_{n \rightarrow \infty} g_n(x_n, z) \in G(x, z)$ for all $z$. Since each $g_n(x_n, \cdot)$ is measurable (in $z$), so is $g(x, z)$ (it is the pointwise limit of the sequence of functions $(g_1(x_1, \cdot), g_2(x_2, \cdot), \ldots)$). Since $f$ is continuous, $f \circ g_n(x_n, \cdot)$ is measurable for all $n$, and so we have:

33Specifically, this is true when $X \times Z$ is a metric space with the Borel algebra and a complete $\sigma$-finite measure (see Aubin and Frankowska (1990) for details and a proof).
\[
\lim_{n \to \infty} \int f(z) P_{g_n}(x_n, dz) = \lim_{n \to \infty} \int f \circ g_n(x_n, z) \mu_z(dz)
\]

Since \( f \circ g_n(x_n, z) \to f \circ g(x, z) \) for all \( z \) (pointwise), it follows by Lebesgue’s Dominated Convergence Theorem that:

\[
\lim_{n \to \infty} \int f \circ g_n(x_n, z) \mu_z(dz) = \int f \circ g(x, z) \mu_z(dz)
\]

Combining the above expressions we conclude that \( \lim_{n \to \infty} \int f P_{g_n}(x_n, dz) = \int f \circ g(x, z) \mu_z(dz) = \int f(z) P_g(x, dz) \) which is what we wanted to show. ■

**Remark 7** Since an upper hemi-continuous correspondence is measurable, we get what Blume (1982) calls a multi-valued stochastic kernel \( K : X \to 2^{P(X)} \) by taking \( P(x, \cdot) = K(x) \) for all \( x \in X \).

Given \( g \in \mathcal{G} \), define the adjoint Markov operator in the usual way from the transition function \( P_g \):

\[
T^*_g \lambda = \int P_g(x, \cdot) \lambda(dx)
\]

Next define the **adjoint Markov correspondence**:

\[
T^* \lambda = \{T^*_g \lambda : g \in \mathcal{G}\}
\]

To clarify, \( T^* \) maps a probability measure \( \lambda \) into a set of probability measures, namely the set \( \{T^*_g \lambda : g \in \mathcal{G}\} \). A probability measure \( \lambda^* \) is **invariant** if:

\[
\lambda^* \in T^* \lambda^*
\]

Of course this is the same as saying that there exists \( g \in \mathcal{G} \) such that \( \lambda^* = T^*_g \lambda^* \).

**Lemma 3** (The Adjoint Markov Correspondence is Upper Hemi-Continuous) Let \( \lambda_n \to_w \lambda \) and consider a sequence \((\mu_n)\) with \( \mu_n \in T^* \lambda_n \). Then for any convergent subsequence \( \mu_{n_m} \to_w \mu \), it holds that \( \mu \in T^* \lambda \).

**Proof.** Although easy to prove directly, we shall not because it is a direct consequence of Proposition 2.3. in Blume (1982) (see Remark 7). ■

One way to prove existence of an invariant distribution with transition correspondences is based on convexity, upper hemi-continuity, and the Kakutani-Glicksberg-Fan Theorem (Blume...
Alternatively, one can look at suitable increasing selections and prove existence along the lines of Hopenhayn and Prescott (1992) using the Knaster-Tarski Theorem. However, for this paper’s developments, we need a set-valued existence result that integrates with the results of Section ???. Mathematically, this can be accomplished by using the set-valued fixed point theorem of Smithson (1971), and this is what we shall do below. The order is the first-order stochastic dominance order.

We begin by proving a new result saying that if the policy correspondence \( G(x, z) \) has an increasing and measurable greatest (respectively, least) selection in \( x \) (for fixed \( z \)), then the adjoint Markov correspondence will be type I (respectively, type II) monotone in the sense of Definition 4.

**Theorem 11** Assume that the policy correspondence \( G : X \times \{z\} \to 2^X \) has an increasing greatest [least] selection for each fixed \( z \in Z \). Then the adjoint Markov correspondence \( T^* \) is type I [type II] monotone. If \( G \) depends on an exogenous variable \( a \in A \) so that \( G : X \times \{z\} \times A \to 2^X \) and the greatest [least] selection from \( G \) is increasing in \( a \), then \( T^*_a \) will in addition be type I [type II] monotone in \( a \).

**Proof.** We prove the greatest/type I case only (the second case is similar). Consider probability measures \( \mu_2 \succneq \mu_1 \). We wish to show that for any \( \lambda_1 \in T^* \mu_1 \), there exists \( \lambda_2 \in T^* \mu_2 \) such that \( \lambda_2 \succeq \lambda_1 \). Let \( \lambda_1 \in T^* \mu_1 \) if and only if there exists a measurable selection \( g_1 \in G \) such that:

\[
\lambda_1(\cdot) = \int_X P_{g_1}(x, \cdot) \mu_1(dx)
\]

where,

\[
P_{g_1}(x, A) = \int_Z \chi_A(g_1(x, z)) \mu_z(dz) , \text{ for } A \in B(X)
\]

Similarly for \( \lambda_2 \in T^* \mu_2 \) where we denote the (not yet determined) measurable selection by \( g_2 \in G \). Given these measurable selections, we have \( \lambda_2 \succeq \lambda_1 \) if and only if for every increasing function \( f \):

\[
\int_X f(x) \lambda_2(dx) \geq \int_X f(x) \lambda_1(dx) \Leftrightarrow
\]

\[
\int_X \int_Z f \circ g_2(x, z) \mu_z(dz) \mu_2(dx) \geq \int_X \int_Z f \circ g_1(x, z) \mu_z(dz) \mu_1(dx)
\] (24)

But taking \( g_2 \) to be the greatest selection from \( G \) (which is measurable), it is clear that,

\[
\int_X \int_Z f \circ g_2(x, z) \mu_z(dz) \mu_1(dx) \geq \int_X \int_Z f \circ g_1(x, z) \mu_z(dz) \mu_1(dx)
\] (25)

39
In addition, since $g_2$ is increasing in $x$, the function $x \mapsto \int_Z f \circ g_2(x, z) \mu_z(dz)$ is increasing in $x$. Since $\mu_2 \succeq \mu_1$ it follows that,

$$\int_X \int_Z f \circ g_2(x, z) \mu_z(dz) \mu_2(dx) \geq \int_X \int_Z f \circ g_2(x, z) \mu_z(dz) \mu_1(dx)$$

(26)

Now simply combine (25) and (26) to get (24). Thus we have proved that if $G$ has an increasing maximal selection, $T^*$ will be type I monotone.

The statements concerning the variable $a \in A$ are proved by essentially the same argument and is omitted. ■

We now get the following existence result. Note that unless $T^*$ is also convex valued (which is not assumed here), the set of invariant distributions will generally not be convex.

**Theorem 12 (Existence in the Type I/II Monotone Case)** Assume that the adjoint Markov correspondence is either type I (or type II) order preserving. In addition assume that the state space (strategy set) has an infimum. Then $T^*$ has a fixed point (there exists an invariant measure). In addition, the fixed point correspondence will be upper hemi-continuous if $T^*$ is upper hemi-continuous in $(\mu, \theta)$ where $\theta$ is a parameter.

**Proof.** By Proposition 1 in Hopenhayn and Prescott (1992), $(\mathcal{P}(X), \succeq)$ is chain complete (meaning that any chain $C$ in $\mathcal{P}(X)$ has a supremum in $\mathcal{P}(X)$). In order to apply Theorem 1.1. of Smithson (1971) we need therefore only verify his “Condition III” and establish the existence of some $\mu \in \mathcal{P}(X)$ such that there exists a $\lambda \in T^* \mu$ with $\mu \preceq \lambda$. The first of these (“Condition III”) follows directly from upper hemi-continuity of $T^*$ (proof omitted). For the second, we do as Hopenhayn and Prescott (1992), proof of Corollary 2, and pick a measure $\delta_a$ from $\mathcal{P}(X)$ that places probability one on the infimum $\{a\} \equiv \inf X \in X$. Then $\lambda \succeq \delta_a$ for all $\lambda \in \mathcal{P}(X)$. It is then clear that if we take $\mu = \delta_a$ we have $\lambda \succeq \mu$ for (in fact, every) $\lambda \in T^* \mu$. The upper hemi-continuity claim is trivial under the stated assumptions. ■

8.3 Appendix III: Aggregation of Risk and Laws of Large Numbers

This appendix is devoted to possible mathematical interpretations of the baseline aggregator (7) of Section 3:

$$H((x_{i,t})_{i\in I}) = \int_{[0,1]} x_{i,t} di$$

(27)

Since the integrands on the right-hand-side of (27) are random variables, we must define what it means to integrate across them. And the fact is that there simply is not a uniformly accepted
way to define this. In addition, we must ensure that some law of large numbers supports the assertion that the function’s values are real numbers. There is a large and growing theoretical literature on how this can be done. The following are some of the most popular approaches to eliminating risk at the aggregate level.

- **(The Sampling Approach)** If one defines the integral \( \int_{[0,1]} x_{i,t} \, di \) as the limiting average over an infinite (randomly drawn) subset of agents (Bewley (1986)), a law of large numbers will immediately apply and \( H \) will take values in \( \mathbb{R} \).

- **(Stochastic Integrals)** Integrals of random functions with respect to deterministic measures is a special case of integrals of random functions with respect to random measures, also known as stochastic integrals.\(^{34}\) Viewing \( \int_{[0,1]} x_i \, di \) as a stochastic integral, we have:

\[
\int_{[0,1]} x_i \, di \equiv \lim_{n \to \infty} \sum_{i=1}^{n} x_{t_i}(t_i - t_{i-1})
\]

(28)

where the convergence is usually taken to be in \( L^2 \)-norm, and as \( n \to \infty \), the lengths of the subdivision \( 0 = t_1 < t_2 < \ldots < t_n = 1 \) tends to zero. Given this interpretation, \( \int_{[0,1]} x_{i,t} \, di \) will itself be a random variable, but when the \( x_i \)’s satisfy assumptions of some appropriate law of large numbers, the distribution will be degenerate (see the Appendix in Acemoglu and Jensen (2010) for further details). We may then identify it with a real number \( H((x_{i,t})_{i \in I}) \) equal to the degenerate distribution’s point of unit-mass as explained above. See also Uhlig (1996) for further details of this approach, including a discussion of its suitability as a description of the limit of an increasing sequence of finite economies.

- **(Pathwise Integration)** Another interpretation of (7) is that of Judd (1985) and Feldman and Gilles (1985) who suggest integrating over the set of sample paths (or rather, the measurable ones). As Judd (1985) and Feldman and Gilles (1985) explain, this approach runs into technical difficulties, however, making it inappropriate for the present purposes.

- **(Discrete Set of Players)** In some contexts it may be unappealing to look at a continuum of agents, but one still wishes formalize the notion that each player is infinitely small relative to the market so that aggregate risk disappears by a law of large numbers type of argument. A way to model this is to look at a countable set of agents \( I \subseteq [0,1] \) (think of an infinitely fine “grid” such as the set of rational numbers) and equip this set with a non-atomic measure.

\(^{34}\)One can think of the former case as a stochastic integral where the random measures being integrated with respect to has a degenerate distribution. See for example Gourieroux (1997), page 71-72, who develops stochastic integrals from precisely this perspective (beginning with the deterministic measure case considered here.)
Such a measure cannot be countably additive (or else the measure of the entire set of players would be 0). The setting thus becomes non-standard, but the advantage is that pathwise integration over sample paths becomes well-defined and the integral over a sample path will equal the sample average almost surely (the difficulties mention in the previous case thus disappear). See Al-Najjar (2004) for more on this idea. In terms of (27), the expression \( \int_{[0,1]} x_{i,t} \, dt \) must now be interpreted as the integral over sample paths. When a law of large numbers applies we once again get a well-defined aggregator. As may be verified, this paper’s results nowhere make explicit use of our favored assumption of a continuum of agents - so if the reader prefers the approach of Al-Najjar (2004) (or any other non-standard approach for that matter), this is easily accommodated by taking \( I \) to be an uncountable but discrete set throughout.

The stochastic integral approach of Uhlig (1996) is further detailed in Acemoglu and Jensen (2010). Here we wish to expand upon this approach. To repeat, the idea is to take \( \int_{[0,1]} x_{i,t} \, dt \) to be equal to the random variable that is given by the limit in \( L^2 \)-norm of the sequence of “Riemann sums”, \( \sum_{i=1}^{n} x_{\tau_i,t}(\tau_i - \tau_{i-1}), n = 1, 2, 3, \ldots \), for a narrowing sequence of subdivisions \( 0 = \tau_1 < \tau_2 < \ldots < \tau_n = 1, n = 1, 2, 3, \ldots \). Whenever the random variables considered are bounded (which they will be in our setting, cf. Assumption 1), convergence in \( L^2 \)-norm is equivalent to convergence in probability.\(^{35}\) Another thing worth mentioning is that sums of the type \( \sum_{i=1}^{n} x_{\tau_i,t}(\tau_i - \tau_{i-1}) \) may seem less general than the standard Riemann sums considered by Uhlig (1996) (precisely, the standard definition of the Riemann integral uses tagged partitions of the type \( \sum_{i=1}^{n} x_{\rho_i,t}(\tau_i - \tau_{i-1}) \) where \( \rho_i \in [\tau_{i-1}, \tau_i] \) for all \( i \). However, it is well known that using the “left-hand” Riemann sum is no less general than general tagged partitions for the simple reason that any tagged partition can be subdivided into a new finer partition whose subdivisions’ left end-points are precisely the original tags. The following lemma is useful for determining when the limit of the Riemann sums is well-defined, and evaluating the integral.

**Lemma 4** Consider the integral in (27) defined as the \( L^2 \)-norm limit of the Riemann sums as described above. Then the limit is well-defined (i.e., it exists and is independent on the subdivisions) if the following condition is met:

\[
\int_{[0,1]} \mathbb{E}[(x_{i,t})^2] \, dt < +\infty \tag{29}
\]

Furthermore, under this condition (or any other condition that implies that the limit is well-defined), the integral can be calculated as:

\(^{35}\)Since convergence almost surely implies convergence in probability, \( L^2 \)-norm convergence is consequently weaker than convergence almost surely in the present setting. This will be used repeatedly below.
\[
\int_{[0,1]} x_{i,t} di = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} X_{i,t}
\]

where \((X_{1,t}, X_{2,t}, X_{3,t}, \ldots)\) is the sequence of random variables defined by recursively halving the interval \([0,1]\), i.e., \(X_{1,t} \equiv x_{1,t} \), \(X_{2,t} \equiv x_{\frac{1}{2},t} \), \(X_{3,t} \equiv x_{\frac{3}{4},t} \), \(X_{4,t} \equiv x_{\frac{3}{8},t} \), \(X_{5,t} \equiv x_{\frac{7}{8},t} \), \ldots.

**Proof.** The first claim of the lemma is found in Gourieroux (1997), p.71. As for (30), we begin by observing that when the subdivisions do not matter, we may focus attention on a convenient sequence of subdivisions such as the even subdivisions, \(0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < 1, \ n = 1, 2, 3, \ldots\). With this subdivision, (27) becomes:

\[
\int_{[0,1]} x_{i,t} di = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} x_{\frac{i}{n},t}
\]

In the summation we get for a given \(n\) a sum over the set of random variables \(\{x_{\frac{1}{n},t}, x_{\frac{2}{n},t}, \ldots, x_{\frac{n-1}{n},t}, x_{1,t}\}\). If we look at the sequence at the subsequence \(n = 1, 2, 4, 8, \ldots\) (which we clearly may do without loss of generality), we get an expanding sequence of random variables: \(\{x_{1,t}\} \subseteq \{x_{\frac{1}{2},t}, x_{1,t}\} \subseteq \{x_{\frac{1}{4},t}, x_{\frac{3}{4},t}, x_{1,t}\} \subseteq \{x_{\frac{1}{8},t}, x_{\frac{3}{8},t}, x_{\frac{5}{8},t}, x_{\frac{7}{8},t}, x_{1,t}\} \subseteq \ldots\). In terms of (31) (slightly modified to the subsequence \(n = 1, 2, 4, \ldots\)), this exactly brings us to the sequence of random variables \(X_{1,t}, X_{2,t}, X_{3,t}, \ldots\).

The upshot of the previous lemma is that (30) allows us to appeal to a standard version of the law of large numbers such as that of Chebyshev (1867) (see Acemoglu and Jensen (2010)) in order to conclude that:

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} X_i
\]

will be a degenerate random variable with unit mass at:

\[
\lim_{n \to \infty} E[\int \sum_{i=1}^{n} \frac{1}{n} X_i]
\]

The only remaining problem then is to ensure that this limit exists.

**Lemma 5** If the function \(i \mapsto E[X_i]\) is Riemann-integrable, \(\mu_n\) converges to the limit \(\int E[X_i] di\) as \(n \to \infty\) where the integral is the Riemann integral.\(^{36}\)

\(^{36}\)When the Riemann integral exists, as it does here by assumption, the Riemann and Lebesgue integrals coincide. So it would be equally true to write that \(\mu_n \to \int E[X_i] di\) where the integral is the Lebesgue integral.
Proof. Since $\mu_n = E[A_n] = \sum_{i=1}^{n} \frac{1}{n} E[X_i]$ is a Riemann sum, the existence of a limit follows directly from the definition of the Riemann integral.

If there is an at most countable number of types, the previous lemma applies. This is because the function $i \mapsto E[X_i]$ will in this case be continuous almost everywhere (in fact, it will be piecewise constant). Since a bounded and continuous almost everywhere function is Riemann integrable, the conclusion follows.

References


