Power, ideology, and electoral competition

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December 2009
Problem

The literature on electoral competition frequently assumes that politicians are single-minded and symmetric:

(A1) They care about either power, or ideology;
(A2) They possess identical electoral motives.

Experimental data doesn’t seem to support (A1):

- Morton (APSR 1993) found subjects in the lab placed a weight of approximately 32% on winning the election, and 68% on the expected payoffs from policies.

Here we relax both (A1) and (A2):

- We study a one-dimensional model of electoral competition with two candidates, which care about winning the election, but also about the policy implemented after the contest, and not necessarily in the same way.
Formal models of electoral competition begin with the famous location model of Hotelling (EJ 1929).

The main results in the one-dimensional case are as follows:

- Certainty-Policy
- Uncertainty-Policy
- Certainty-Office
- Uncertainty-Office

Information about voters' preferences
Parties' motivations
Median ideal policy
Policy differentiation
Expected median ideal policy

Hybrid electoral game under uncertainty

**Figure:** Electoral competition with homogeneous motives
Certainty - winning the election:

- There is a unique NE in which both candidates propose the same policy (median ideal point) – Hotelling (EJ 1929), Downs (1957).

Uncertainty - winning the election:

- **Common prior**: candidates converge to the expected median ideal point – Calvert (AJPS 1985);

- **Private polling**: equilibrium policies diverge, but less than voters would like; candidates with moderate (extreme) signals choose more extreme (moderate) policies than their information recommends; policy convergence doesn’t maximize voters’ welfare – Bernhardt et al. (JET 2009).
Literature

Certainty - winning policy:

► Candidates converge to the median – Calvert (AJPS 1985), Roemer (SCW 1994), Duggan and Fey (GEB 2005).

Uncertainty - winning policy:

► The election game has a PSE, but equilibrium platforms do not necessarily converge – Roemer (SCW 1997, 2001);

► Experimental data supports that uncertainty over voters’ preferences is a major determinant of platform divergence when candidates are ideological – Morton (APSR 1993).
Literature

Certainty - mixed motivations:

▶ There is a unique PSE in which platforms converge to the median ideal point – Calvert (AJPS 1985);

▶ Continuity result: Small departures from “certainty - office motivation” lead to small changes in the equilibrium platforms – Calvert (AJPS 1985).

Uncertainty - mixed motivations:

▶ Payoffs’ discontinuities created by MMA ⇒ PSE might not exist; instead MSE always exists – Ball (SCW 1999);

▶ If candidates possess mixed but homogeneous interests, there is a unique PSE – Saporiti (JPET 2008);

▶ Regardless of candidates’ motives, the mixed extension is b.r.s.; thereby, a NE always exists – Saporiti (JPET 2008);

▶ Stochastic preferences: policy differentiation enhances voters’ welfare – Bernhardt et al. (APSR 2009).
Two candidates, indexed by $i = L, R$, compete in a winner-take-all election by simultaneously announcing a platform $x_i \in X = [0, 1]$.

The electorate is made of a continuum of voters.

Each voter is endowed with a type $\theta$, which is a random draw from the uniform distribution $U$ over $[0, 1]$.

An individual of type $\theta$ has preferences over $X$ represented by the utility function $u_\theta(x) = -w(|x - \theta|)$, where $| \cdot |$ denotes the absolute value on $\mathbb{R}$, and $w$ is a twice differentiable real-valued function, with $w'(0) = 0$, and $w'(z) > 0$ and $w''(z) \geq 0 \forall z > 0$.

Each voter votes sincerely for the platform it likes the most.
A platform wins the election if it gets more than half of the votes; otherwise, each candidate’s proposal wins with probability 1/2.

Besides the uncertainty due to the possibility of a tie, candidates have uncertainty about voters’ preferences.

They perceive the fraction of types supporting their respective platforms with a noise \( \xi \), which is uniformly distributed over the interval \([-\beta, \beta]\), with \( \beta > 0 \) – Roemer’s (2001) error model.

This “noise” is meant to reflect that voters’ preferences and their participation rate are usually hard to predict.
Setup

The uncertainty model adopted above implies that, for any strategy profile \((x_L, x_R) \in X^2\), the probability that candidate \(L\) attaches to winning the election is given by

\[
p(x_L, x_R) = \text{Prob}\left\{ U[S(x_L, x_R)] + \xi > \frac{1}{2} \right\}.
\]

Candidate \(R\)’s probability of winning is given by \(1 - p(x_L, x_R)\).

- If \(x_L = x_R\), then \(p(x_L, x_R) = \frac{1}{2}\).

- Otherwise, if \(x_L \neq x_R\), then:

(i) \(p(x_L, x_R) = \frac{1}{2} + \frac{U[S(x_L, x_R)] - \frac{1}{2}}{2\beta}\) if \(\frac{1}{2} - U[S(x_L, x_R)] \in (-\beta, \beta)\);

(ii) whereas \(p(x_L, x_R) = 1\) (resp., \(p(x_L, x_R) = 0\)) if \(\frac{1}{2} - U[S(x_L, x_R)] \leq -\beta\) (resp., \(\frac{1}{2} - U[S(x_L, x_R)] \geq \beta\)).
Setup

\[ p(x_L, x_R) = 1 \]

\[ p(x_L, x_R) = 0 \]

\[ p(x_L, x_R) = \frac{1}{2} + \frac{1-x_L-x_R}{4\beta} \]

\[ p(x_L, x_R) = \frac{1}{2} - \frac{1-x_L-x_R}{4\beta} \]

**Figure:** Probability of winning
Lemma (1)

For any two platforms $x_L < x_R$ (resp., $x_L > x_R$), $p(x_L, x_R)$ is non-decreasing (resp., non-increasing) in $x_i$, for all $i = L, R$.

Lemma 1 reflects the spatial nature of electoral competition:

- If one candidate moves its platform toward that of its opponent, then it does not decrease (and may increase) the probability with which it wins the election;

- If it moves its platform away from its opponent’s, then it does not increase (and may decrease) its probability of winning.
Setup

Candidates possess mixed or hybrid motives for running for office:

- They are office-motivated, in the sense that they intrinsically value winning the election, and
- They are policy-motivated too, because they care about what policy is enacted after the election.

Formally, the payoffs associated with any pair \((x_L, x_R) \in X^2\) are:

\[
\pi_L(x_L, x_R) = p(x_L, x_R) \cdot [\psi_{\theta_L}(x_L, x_R) + \chi_L],
\]

and

\[
\pi_R(x_L, x_R) = (1 - p(x_L, x_R)) \cdot [\psi_{\theta_R}(x_R, x_L) + \chi_R],
\]

where \(\chi_i > 0\) denotes candidate \(i\)’s intrinsic value (rents) for being in power; \(\theta_i\) stands for candidate \(i\)’s ideological position on \(X\); and for any pair \((x, y) \in X^2\), \(\psi_{\theta}(x, y) = u_{\theta}(x) - u_{\theta}(y)\).
Setup

N.B. Strictly speaking, candidate $L$’s payoff function is
\[
\Pi_L(x_L, x_R) = p(x_L, x_R) \cdot (u_{\theta_L}(x_L) + \chi_L) + [1 - p(x_L, x_R)] \cdot u_{\theta_L}(x_R).
\]
However, since $u_{\theta_L}(x_R)$ doesn’t affect $L$’s optimal choices, we work with the linear transformation $\pi_L(x_L, x_L) \equiv \Pi_L(x_L, x_R) - u_{\theta_L}(x_R)$.

In the sequel, it is assumed that

- $\theta_L < 1/2 < \theta_R$;
- $\beta < \min \left\{ \frac{1}{2} - \theta_L + \frac{\chi_L}{2}, \theta_R - \frac{1}{2} + \frac{\chi_R}{2} \right\}$;
- Candidates are risk neutral.

N.B. Risk neutrality entails a loss of generality because, in spite of being ideologically different, risk averse candidates tend to move closer to each other and toward to the center.

Call $G = (X, \pi_i)_{i=L,R}$ the hybrid election game described before.
Suppose candidates are allowed to randomize over platforms.

Let $\Delta$ be the space of all probability measures on $(X, \mathcal{A})$, where $\mathcal{A}$ is the Borel sigma-algebra on $X$.

A mixed strategy for $i$ is a probability measure $\mu_i \in \Delta$.

For every $(\mu_L, \mu_R) \in \Delta^2$, candidate $i$’s expected payoff is defined as

$$U_i(\mu_L, \mu_R) = \int_{X^2} \pi_i(x_L, x_R) \, d(\mu_L(x_L) \times \mu_R(x_R)).$$

$U_i$ is well defined because $\pi_i$ is (Borel) measurable on $X^2$ (Saporiti, JPET 2008) – the set of discontinuities has measure zero.

The mixed extension of $G$ is given by $\overline{G} = (\Delta, U_i)_{i=L,R}$. 
A Nash equilibrium (NE) for $\overline{G}$ is a pair of probability measures $(\mu^*_L, \mu^*_R) \in \Delta^2$ such that for all $(x_L, x_R) \in X^2$,

- $U_L(\mu^*_L, \mu^*_R) \geq U_L(x_L, \mu^*_R)$, and
- $U_R(\mu^*_L, \mu^*_R) \geq U_R(\mu^*_L, x_R)$.

Let $(\mu^*_L, \mu^*_R) \in \Delta^2$ be a NE of $\overline{G}$:

- If for all $i = L, R$, supp$(\mu^*_i) = \{x^*_i\}$ for some $x^*_i \in X$, the profile $(\mu^*_L, \mu^*_R)$ is said to be a PSE of $\overline{G} = (X, \pi_i)_{i=L,R}$;
- If one or more candidates randomize over two or more policies, $(\mu^*_L, \mu^*_R)$ is said to be a MSE of $\overline{G} = (X, \pi_i)_{i=L,R}$.
Equilibrium analysis

Proving that $\bar{G}$ possesses a NE is not a trivial matter because $G$ is an infinite action game with discontinuous payoffs.

Saporiti (JPET 2008) has shown that

1. If $\chi_L = \chi_R$ (homogenous interests), then $G$ has a unique PSE;

2. If $\chi_L \neq \chi_R$ (heterogeneous interests), then depending upon the relationship between the electoral uncertainty ($\beta$), the aggregate interest in office ($\chi_L + \chi_R$), and its distribution across the candidates, a NE in pure strategies may not exist;

3. $\bar{G}$ satisfies Reny’s (ECTA 1999) better reply security; therefore a NE always exists.

The results in (1) and (3) hold for any probability distribution of $\xi$ and $\theta$, provided that they are both continuous over the support.
Equilibrium analysis

While the existence result is interesting, it falls short of providing any prediction about the location of equilibrium platforms.

To take a step forward, in what follows we study the set of NE of the hybrid election game introduced above.

We begin by observing that $G$ possesses neither (i) a PSE where $L$ chooses a platform further to the right than $R$’s proposal, nor (ii) a PSE where one of the candidates wins the election for sure.

Lemma (2)

If the strategy profile $(x_L^*, x_R^*) \in X^2$ is a PSE for the election game $G = (X, \pi_i)_{i=L,R}$, then $\theta_L < x_L^* \leq x_R^* < \theta_R$ and $p(x_L^*, x_R^*) \in (0, 1)$.
Equilibrium analysis

Lemma 2 allows us to focus the equilibrium analysis on the white and red regions of Figure 2. It is used below to characterize each candidate’s platform in a PSE with policy differentiation, and to provide a necessary condition for such equilibrium to exist.

Lemma (3)

The election game $G = (X, \pi_i)_{i=L,R}$ has a PSE with $x^*_L < x^*_R$ only if $\chi_L + \chi_R < 4\beta$, $x^*_L = \frac{1}{2} - \beta + \frac{\chi_L}{2}$, and $x^*_R = \frac{1}{2} + \beta - \frac{\chi_R}{2}$.

The platforms characterized in Lemma 3 are a function of the uncertainty $\beta$ and the office rents $\chi_i$, and with the expected sign:

- All the rest equal, a reduction of the electoral uncertainty (respectively, an increase in the office rents) moves both platforms toward to the center.
Equilibrium analysis

These policies, however, are independent of the candidates’ ideological positions.

And they are independent of each other too, in the sense that a change in candidate $i$’s equilibrium policy $x_i^*$ (due for example to a change in $\chi_i$), does not affect $x_j^*$.

These are mainly consequences of risk neutrality:

- If candidates are risk averse, it is easy to construct examples where equilibrium platforms are interdependent and sensible (directly or indirectly) to the ideology of each candidate.
Equilibrium analysis

The next two propositions provide sufficient conditions for two types of platform configurations that can emerge from Lemma 3.

Proposition (two-sided policy differentiation)

The election game $\mathcal{G} = (X, \pi_i)_{i=L,R}$ has a PSE with $x^*_L < \frac{1}{2} < x^*_R$ if $\chi_i < 2\beta$ for all $i = L, R$.

Proposition (one-sided policy differentiation)

The election game $\mathcal{G} = (X, \pi_i)_{i=L,R}$ has a PSE with $\frac{1}{2} < x^*_L < x^*_R$ if (i) $\chi_L > 2\beta$, (ii) $\chi_L + \chi_R < 4\beta$, and (iii) $\chi_L \cdot \chi_R \leq 4\beta^2 + (\chi_L - \chi_R)^2/4 - 2\beta(\chi_L - \chi_R)$.

N.B. A similar statement can be formulated for $\chi_R > 2\beta$, though in that case candidates place their platforms over the left-hand side of the median voter’s preferred policy; i.e. $x^*_L < x^*_R < \frac{1}{2}$. 
Equilibrium analysis

Example: Suppose $\chi_L = 0.6$, $\chi_R = 0.05$, and $\beta = 0.25$. For these values, $x_L^* = 0.55$ and $x_R^* = 0.725$. The graphs below confirm that this pair is a PSE with one-sided differentiation.

Figure: $L$’s conditional payoff given $x_R^* = 0.725$.

Figure: $R$’s conditional payoff given $x_L^* = 0.55$. 
Equilibrium analysis

Condition (iii) in Proposition 2 rules out situations like the one presented below (for $\chi_L = 0.6$, $\chi_R = 0.2$, and $\beta = 0.25$).

\[ \Pi_R /\left(1 x_L /, \cdot /\right) x_R /\star /\text{equal} 0.65 \]

Figure: $R$’s conditional payoff given $x_L^* = 0.55$. 
Equilibrium analysis

As a matter of illustration, let us point out that evidence of one-sided differentiation over the center-right appears to be found in the British election of the year 1997.

From a total of 659 parliamentary seats contested in 1997, the Conservative and Labour party got together 88.7% of them;

▶ Roughly a two-candidate election.

All policy dimensions with relatively high salience in the 1997 electoral contest were highly correlated with the economic dimension (Laver, PS 1998);

▶ Roughly a one-dimensional election.

The Labour (L) and the Conservative (C) party policy positions were as follows:
The positions shown in the graph are based on a survey conducted among 117 political scientists in British universities.

Alternative techniques (namely, hand-coded, computer-coded, and word-scoring analysis of political texts) provide similar results – Laver and Garry (AJPS 2000) and Laver et al. (APSR 2003).

The graph also seems to show two-sided differentiation in the 1989 election; however, we didn’t check if that was in fact the case.
Equilibrium analysis

What about policy convergence? When does it emerge?

Proposition (policy convergence)

The election game $G = (X, \pi_i)_{i=L,R}$ has a PSE with $x^*_L = x^*_R \equiv x^*$ if and only if $x^* = \frac{1}{2}$ and $\chi_i \geq 2\beta$ for all $i = L, R$.

Proposition 3 resembles Calvert’s (AJPS 1985) continuity result.

It asserts that both candidates are going to choose a platform located over the expected median ideal point if and only if the relative value of holding office $\chi_i/2\beta$ is high enough for all $i$.

N.B. $\chi_i/2\beta$ is a kind of adjusted or relative value of office:

- A winning platform offers a payoff $\chi_i$ for being in office;
- But hitting the median and winning the election with a particular policy has a chance of $(2\beta)^{-1}$ (the inverse of the length of the support of the error term $\xi$).
Equilibrium analysis

An obvious implication of Proposition 3 and Lemma 3 is as follows.

**Corollary (uniqueness)**

If the election game $G = (X, \pi_i)_{i=L,R}$ possesses a pure strategy equilibrium, then the equilibrium is unique.

We offer now some insights about the welfare properties of the equilibria characterized above.

For any pair $(x_L, x_R) \in X^2$, denote by

$$V_\theta(x_L, x_R) = p(x_L, x_R) \cdot u_\theta(x_L) + [1 - p(x_L, x_R)] \cdot u_\theta(x_R)$$

the expected welfare of a voter of type $\theta \in [0, 1]$. 

Equilibrium analysis

Proposition (social welfare)

Suppose voters are risk averse ($w'' > 0$) and $\chi_L + \chi_R < 4\beta$. Let $x_L^* = \frac{1}{2} - \beta + \frac{\chi_L}{2}$ and $x_R^* = \frac{1}{2} + \beta - \frac{\chi_R}{2}$ be equilibrium policies. Then:

(i) If $\chi_L = \chi_R$, for all $\theta \in [0, 1]$, $V_\theta(x_L^*, x_R^*) < V_\theta(\frac{1}{2}, \frac{1}{2})$; and

(ii) If $\chi_L > \chi_R$ ($\chi_L < \chi_R$), there exists $\hat{\theta} > \frac{1}{2}$ ($\hat{\theta} < \frac{1}{2}$) such that for all $\theta \leq \hat{\theta}$ ($\theta \geq \hat{\theta}$), $V_\theta(x_L^*, x_R^*) < V_\theta(\frac{1}{2}, \frac{1}{2})$.

In contrast with Bernhardt et al. (APSR 2009), Prop. 4 shows that, when the source of the electoral uncertainty is the lack of perfectly accurate assessments about voters’ preferences, rather than a shock over their policy positions, policy convergence is preferred to policy differentiation by a strict majority.
Equilibrium analysis

From the previous analysis, it is easy to see that the example behind Fig. 5, where $\chi_L = 0.6$, $\chi_R = 0.2$, and $\beta = 0.25$, doesn’t admit a PSE:

1. By Lemma 2, if $\exists$ a PSE, it must be that $x_L \leq x_R$;

2. $\not\exists$ PSE with $x_L < x_R$ because Lemma 3 requires $x_L = 0.55$ and $x_R = 0.65$, but Fig. 5 shows $\pi_R(0.55, x_R)$ isn’t maximized at $x_R = 0.65$;

3. Thus, by Prop. 3, the only possibility is $(\frac{1}{2}, \frac{1}{2})$;

4. A deviation $x'_i = \frac{1}{2} \pm \delta$ from $x_i = \frac{1}{2}$ is profitable if $\delta < 2\beta - \chi_i$. Therefore, $(\frac{1}{2}, \frac{1}{2})$ is not a PSE either, since $\pi_R(\frac{1}{2}, x'_R) > \pi_R(\frac{1}{2}, \frac{1}{2})$ for any $\delta < 0.3$. 
Equilibrium analysis

Fortunately, we know a MSE does exist. The next result sheds some light on the properties of those equilibria.

For every $x \in [0, 1]$ and any $\mu \in \Delta$, the c.d.f. $G$ at $x$ associated with $\mu$ is defined by $G(x) = \mu([0, x])$.

Proposition (probabilistic differentiation)

The election game $G = (X, \pi_i)_{i=L,R}$ has a MSE if (i) $\kappa_i < 2\beta$ for some $i = L, R$, and (ii) $\kappa_L + \kappa_R \geq 4\beta$. If $(\mu^*_L, \mu^*_R) \in \Delta^2$ is a MSE of $G$ satisfying conditions (i) and (ii) and $G^*_i$ denotes the c.d.f. associated with $\mu^*_i$, then for all $i = L, R$, 

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Equilibrium analysis

1. If $\chi_L < 2\beta$, supp($\mu_i^*$) = $[x, \bar{x}]$, with $x = \frac{1}{2} - \beta + \frac{\chi_L}{2}$ & $\bar{x} = \frac{1}{2}$;

2. If $\chi_R < 2\beta$, supp($\mu_i^*$) = $[\underline{x}, \bar{x}]$, with $\underline{x} = \frac{1}{2}$ & $\bar{x} = \frac{1}{2} + \beta - \frac{\chi_R}{2}$;

3. $\mu_i^*$ converges weakly to the point mass on $\frac{1}{2}$ as $\chi_i \to 2\beta$;

4. $G^*_L$ has an atom on $\underline{x}$, i.e. $G^*_L(\underline{x}) - \lim_{x \to \underline{x}^-} G^*_L(x) > 0$;

5. $G^*_R$ has an atom on $\bar{x}$, i.e. $G^*_R(\bar{x}) - \lim_{x \to \bar{x}^-} G^*_R(x) > 0$;

6. $G^*_i$ is atomless and differentiable on $(\underline{x}, \bar{x})$, with density $g_i^*$;

7. $g_i^*$ is U-shaped (Conjecture).

N.B. A MSE with similar properties may also exist if (i) $\chi_i > 2\beta$ for some $i = L, R$, and (ii) $\chi_L + \chi_R < 4\beta$. 
Equilibrium analysis

Summarizing, we could say that in the two-candidate, one-dimensional electoral competition game with mixed motives,

1. The equilibrium platforms are shaped by the electoral uncertainty, candidates’ ideologies, the aggregate interest in office and its distribution across the candidates.

2. If the value of being in power is the same for the two candidates, both announce a policy located at either

   (2.i) The estimated median (convergence), if the uncertainty is low compared with the aggregate interest in office, or

   (2.ii) Over their own ideological side (two-sided differentiation) if the uncertainty is high.

3. If candidates have asymmetric motivations,
Equilibrium analysis

(3.i) Platforms still converge to the estimated median voter’s ideal point for low levels of uncertainty;

(3.ii) By contrast, when the uncertainty increases, a PSE does not always exist, and the hybrid election game possesses a MSE;

(3.iii) The equilibrium support is an interval on one side of the estimated median, and is the same for both candidates;

(3.iv) Probability distributions are atomless in the interior and they accumulate positive mass over the bounds;

(3.v) MSE vanishes above a critical level of uncertainty, beyond which each candidate assigns all of the mass to a different platform, which are for a while over the same ideological side.
Equilibrium analysis

\[ \beta \bar{\beta} \]

\[ C = \chi \]

\[ R \]

\[ 0.5 \]

\[ x^*_i \]

\[ 0.5 \]

\[ 0 \]

\[ \beta_C = \frac{\chi_R}{2} \]

\[ \bar{\beta} \]

\[ \beta \]

Convergence Two-sided differentiation

Figure: Symmetric case: \( \chi_L = \chi_R \).
Equilibrium analysis

\[
\begin{align*}
\beta_C &= \chi_R / 2; \\
\beta_C' &= \chi_L + \chi_R / 4; \\
\beta_C'' &= \chi_L / 2
\end{align*}
\]

Figure: Asymmetric case: \( \chi_L > \chi_R \).
To characterize the properties of the MSE listed above, we’d like to approximate the election game with continuous strategy space by a sequence of successively finer finite games.

However, since $\pi_L + \pi_R$ is not u.s.c. (unless $\chi_L = \chi_R$), we can’t be sure the sequence of MSE of the finite games will converge to an equilibrium of the limit game.

To circumvent this difficulty, we proceed as follows.

First, we show that the payoff functions exhibit complementary discontinuities, in the sense that whenever the payoff of one candidate jumps down, the other jumps up.
About the proof of Prop. 5

Lemma (4)

For every \( y \in X \), there exists \( i, j \in \{L, R\} \), with \( i \neq j \), such that

\[
\lim_{x_L \to -y} \lim_{x_R \to +y} \pi_i(x_L, x_R) \geq \frac{\chi_i}{2} \geq \lim_{x_L \to +y} \lim_{x_R \to -y} \pi_i(x_L, x_R),
\]

(1)

and

\[
\lim_{x_L \to -y} \lim_{x_R \to +y} \pi_j(x_L, x_R) \leq \frac{\chi_j}{2} \leq \lim_{x_L \to +y} \lim_{x_R \to -y} \pi_j(x_L, x_R),
\]

(2)

where the left (resp. right) inequality in (1) is strict if and only if the right (resp. left) inequality in (2) is strict.
About the proof of Prop. 5

Second, we use Lemma 4 to construct an auxiliary (upper semi-continuous sum) game $G_\alpha = (X, \pi_i^\alpha)_{i=L,R}$, where

$$\forall \ell = L, R, \forall x_L \neq x_R, \pi_\ell^\alpha(x_L, x_R) = \pi_\ell(x_L, x_R);$$

and, if $x_L = x_R \equiv y$ for some $y \in X$, then

$$\pi_i^\alpha(y, y) = \begin{cases} \pi_i(y, y) & \text{if the left ineq. in (1) holds with eq. for } i, \\ \alpha_i(y) & \text{if the left ineq. in (1) holds strictly for } i, \end{cases}$$

$$\pi_j^\alpha(y, y) = \begin{cases} \pi_j(y, y) & \text{if the right ineq. in (2) holds with eq. for } j, \\ \alpha_j(y) & \text{if the right ineq. in (2) holds strictly for } j, \end{cases}$$

where $\alpha_i(y)$ and $\alpha_j(y)$ are two “arbitrarily chosen” real numbers.
About the proof of Prop. 5

Third, we prove the following two lemmas:

Lemma (5)

For every \( y \in X \), \( \pi_L^\alpha + \pi_R^\alpha \) is upper semi-continuous at \((y, y)\).

Lemma (6)

For all \( i = L, R \), the payoff function \( \pi_i^\alpha \) is weakly lower semi-continuous in \( x_i \).

Thus, from Theorem 5 in Dasgupta and Maskin (RES 1986), we conclude that \( G_\alpha \) possesses a MSE. Moreover, we show that:

Lemma (7)

A profile \( (\hat{\mu}_L, \hat{\mu}_R) \in \Delta^2 \) is a MSE for \( G_\alpha = (X, \pi_i^\alpha)_{i=L,R} \) if and only if \( (\hat{\mu}_L, \hat{\mu}_R) \) is a MSE for the original game \( G = (X, \pi_i)_{i=L,R} \).
Fourth, we approximate $G_{\alpha} = (X, \pi^\alpha_i)_{i=L,R}$ by a sequence of successively finer finite games $\{G^n_{\alpha}\}_{n>0}$, with

$$G^n_{\alpha} = (X_n, \pi^\alpha_i)_{i=L,R},$$

where $X_n$ denotes $n$-th finite approximation (lattice) of $X$, with $X_n \to X$ as $n \to \infty$, and $\pi^\alpha_i$ in $G^n_{\alpha}$ is the restriction on $X_n \times X_n$.

By the Nash’s theorem, each $G^n_{\alpha}$ possesses a MSE $\mu^n = (\mu^n_L, \mu^n_R)$.

Moreover, the properties of $G_{\alpha}$ ensures that a subsequence of $\{\mu^n\}_n$ converges to a profile $\mu^* = (\mu^*_L, \mu^*_R)$, which is a MSE for $G$.

Finally, the properties of the limit equilibrium $\mu^* = (\mu^*_L, \mu^*_R)$ are proven investigating the properties of the sequence $\{\mu^n\}_n$. 
Stochastic preferences

Instead of the error distribution model of electoral uncertainty studied above, consider the following alternative.

Suppose the distribution of ideal policies within the electorate is known up to a shift parameter, $\xi$, which is a common shock to the environment (e.g., a terrorist attack, a financial crisis, etc.).

Each voter $v$ is initially endowed with a preferred policy $\theta_v$, which is commonly known and randomly drawn from the uniform distribution over $[\beta, 1 - \beta]$, with $0 < \beta < 0.5$.

Voters’ ideal policies are subject to a common and random shock $\xi$, which is distributed uniformly over $[-\beta, \beta]$, and is unobserved by the candidates when they choose their platforms.
**Stochastic preferences**

Let \( \tilde{\theta}_v = \theta_v + \xi \) be voter's \( v \) actual ideal policy, and suppose its utility function over the policy space \( X = [0, 1] \) is as before:

\[
u_v(x) = -w(|x - \tilde{\theta}_v|).
\]

It is easy to see that the median voter’s ideal policy \( \tilde{\theta}_m \) is actually a random variable:

\[
\tilde{\theta}_m \sim U \left[ \frac{1}{2} - \beta, \frac{1}{2} + \beta \right].
\]

Hence, for any pair of platforms \((x_L, x_R) \in X^2\), the left-wing candidate's probability of winning the election is given by

\[
\tilde{p}(x_L, x_R) = \begin{cases} 
\text{Prob}\left( \tilde{\theta}_m \in \left[ 0, \frac{x_L + x_R}{2} \right] \right) & \text{if } x_L \leq x_R, \\
\text{Prob}\left( \tilde{\theta}_m \in \left[ \frac{x_L + x_R}{2}, 1 \right] \right) & \text{if } x_L > x_R.
\end{cases}
\]
It can be shown that the probability of winning function $\tilde{p}(\cdot)$ of this model (with stochastic preferences) coincides with the probability of winning function $p(\cdot)$ of the previous model (with fixed preferences, but a noisy vote share).

Therefore, since the two frameworks are the same in all other respects, the equilibrium analysis carried out before also applies to the current case.

The models differ, however, with respect to the welfare effect of policy differentiation.
Stochastic preferences

To see this, suppose $\chi_L = \chi_R$ (homogeneous motives), and define voter $v$’s ex-ante welfare associated with an equilibrium $(x_L, x_R)$ as:

$$\tilde{V}_v(x_L, x_R) = -\frac{1}{2\beta} \cdot \left[ \int_{-\beta}^{0} w(|x_L - \theta_v - \xi|) d\xi + \int_{0}^{\beta} w(|x_R - \theta_v - \xi|) d\xi \right].$$

Then, for all $\theta_v \in [\beta, 1 - \beta]$, $V_v(x_L^*, x_R^*) \geq V_v(1/2, 1/2)$, where $(x_L^*, x_R^*)$ is the equilibrium profile characterized in Proposition 1.
Figure: Voters’ welfare with stochastic preferences ($\beta = .25$ & $\chi = .1$).
Stochastic preferences

\[ \tilde{V}_v(x_L^*, x_R^*) - \tilde{V}_v(1/2, 1/2) \]

Figure: Voters’ welfare with stochastic preferences \((\beta = .25 \& \chi = .1)\).
Stochastic preferences

Summing up, with stochastic preferences, if candidates possess identical motivations, all voters prefer two-sided differentiation to convergence, reversing the welfare result obtained in Prop. 4.

- The micro foundations of the probability of winning matters in electoral competition, especially for welfare analysis.

- Regarding social welfare, policy differentiation (convergence) may be good or bad depending on the model of electoral uncertainty behind the probability of winning function.