Equilibrium Portfolios in the Neoclassical Growth Model

Emilio Espino*

July 11, 2005

Abstract

This paper studies equilibrium portfolios in the traditional neoclassical growth model under uncertainty with heterogeneous agents and dynamically complete markets. Preferences are restricted to be quasi-homothetic. Heterogeneity across agents is due to two reasons. First, agents may have different shares of the representative firm at date 0. Secondly, agents may also have different preferences but only due to different minimum consumption requirements. Whenever this environment displays changing degrees of heterogeneity across agents, the trading strategy of fixed portfolios cannot be optimal in equilibrium. This trading strategy is optimal in stationary endowment economies with dynamically complete markets. Very importantly, our framework can generate changing heterogeneity either if minimum consumption requirements are not zero or if labor income is not zero and the value of human wealth and non-human are linearly independent.

Preliminary. Comments are welcome.

JEL classification: C61, D50, D90, E20, G11.

Keywords: Neoclassical Growth Model, Equilibrium Portfolios, Complete Markets.

*Department of Economics and Finance, Institute for Advanced Studies (IHS), Stumpergasse 56, 1060, Vienna, Austria. E-mail: espino@ihs.ac.at.
1 Introduction

This paper studies equilibrium portfolios in the traditional one-sector neoclassical growth model under uncertainty with heterogeneous agents. There is an aggregate technology to produce the unique consumption good which is either consumed or invested. This technology displays constant returns to scale, is subject to productivity shocks and is operated by a representative firm. Preferences are purposely restricted such that momentary utility functions are quasi-homothetic. This implies that Engel curves are affine linear in lifetime human and non-human wealth. Heterogeneity across agents can arise due to differences in initial wealth and preferences in the following way. First, agents may have different shares of the representative firm at date 0 but they have the same labor income profile. That is, they can differ in their non-human wealth but share the same human wealth. Secondly, agents may also have different preferences within the class mentioned before due to different minimum consumption requirements. We abstract from any kind of friction and, in particular, we assume that markets are dynamically complete. The environment is consequently an extension of Brock and Mirman’s [1972] seminal contribution in the spirit of Chatterjee’s [1994] pioneering work. The main differences with Chatterjee [1994] are that we consider a stochastic environment due to aggregate technology shocks, agents have labor income and minimum consumption requirements can differ among agents.

Recently, the ability to generate nontrivial asset trading of the celebrated Lucas [1978] tree model has been under scrutiny. The original version is silent about the evolution of equilibrium portfolios since they are kept fixed as a direct consequence of studying a representative agent framework. However, Judd, Kubler and Schmedders [2003] (JKS from now on) have carefully studied the Lucas tree model with complete
markets and assumptions that allow for fairly general patterns of heterogeneity across agents. Their surprising finding is that, after some initial rebalancing in short and long-lived assets, agents choose a fixed equilibrium portfolio which is independent of the state of nature. Unarguably, the Lucas tree model is one of the most popular asset pricing models in modern finance theory and this striking feature may cast additional doubts about its full applicability.\footnote{The pioneer work of Mehra and Prescott [1985] has pointed out some puzzling asset pricing implications of the standard Lucas tree model.} JKS conclude that some friction (informational, financial, etc.) must play a significant role in generating nontrivial asset trading in that framework.

Espino and Hintermaier [2005] have questioned the necessity of those frictions and studied instead the production economy proposed by Brock [1982] with heterogeneous agents under dynamically complete markets. Production of the consumption good is carried out by several neoclassical firms subject to idiosyncratic productivity shocks. Analyzing equilibrium asset trading, they show that their environment can generate a nontrivial amount of trading where equilibrium portfolios in general depend upon the aggregate state of the economy. Consequently, the trading strategy of fixed portfolios is not optimal in equilibrium. In their economy, the time dependence of agents’ heterogeneity is determined by the evolution of the distribution of capital across firms. This is mainly due to the fact that agents have different labor productivities across these different production units.

A recent paper by Bossaerts and Zame [2005] has also questioned JKS’s interpretation of their result. They assume that individual endowments are nonstationary even though this last property holds at the aggregate level. In that case they construct examples where equilibrium portfolios are not kept fixed. But after all, assuming away stationarity of individual endowments implies that the distribution of the ag-

1
aggregate endowment is nonstationary and consequently the degree of heterogeneity across agents is naturally changing.

Thus, we understand that the crucial aspect pointed out by Espino and Hintermaier [2005] and, at least implicitly, Bossaerts and Zame [2005] is the following. Whenever the environment under study generates changing degrees of heterogeneity across agents, the trading strategy of fixed portfolios cannot be optimal in equilibrium. The second of these papers simply assumes that a crucial dimension of heterogeneity changes through time. On the other hand, one might suspect that the results in the first of these papers rely on the assumption regarding different labor productivities across firms (what they call *limited labor substitutability*).

This paper steps back to endogeneously generate changing degrees of heterogeneity abstracting from all those "frictions". According to our understanding, we study the simplest extension of the Lucas tree model to a production economy that is able to analyze asset trading: the stochastic one-sector optimal growth model with heterogeneous agents. The analysis shows how to explicitly determine equilibrium portfolios exploiting some features of the preference representations assumed. Very importantly, this framework can generate changing heterogeneity either if minimum consumption requirements are not zero or if labor income is not zero and the value of human wealth and non-human wealth are linearly independent.

In our stationary economy, at date 0 agents can be different due to two reasons. First, they might have different minimum consumption requirements. This kind of heterogeneity is kept fixed as time and uncertainty unfold. Secondly, agents can own different shares of aggregate non-human wealth at date 0, which is given by the initial value of the firm. If equilibrium evolves such that these shares are kept fixed at the initial level, this crucial second dimension of heterogeneity across agents also remains
unchanged. Consequently, agents can trade away all kind of individual risk through fixed portfolios whenever both channels of heterogeneity remains unchanged.

In our framework, each agent’s participation in non-human wealth is not constant in equilibrium either if minimum consumption requirements are not zero or if labor income is not zero and human and non-human wealth are linearly independent. Consider the case where there are no minimum consumption requirements. Under these preference representations, complete markets will imply that each agent owns a fixed share of aggregate total wealth, which includes human and non-human wealth. Both levels of wealth depend on the evolution of the stock of capital and thus equilibrium portfolios will adjust accordingly to keep these shares fixed whenever human and non-human wealth are not linearly dependent.

When there is no labor income but minimum consumption requirements are not zero, the intuition is similar. In this case, agents will keep a fixed fraction of aggregate non-human wealth net of the value of the aggregate minimum requirement. This will imply that equilibrium portfolios cannot be kept fixed and independent of the state of the economy.

In a recent and independent work, Obiols-Homs and Urrutia [2005] study the evolution of asset holdings in a deterministic version of our framework with log preferences. They find sufficient conditions such that the coefficient of variation in assets across agents decreases monotonically along the transition to the steady state from below. In contrast, Chatterjee [1994] studies the evolution of wealth in an economy without labor income. He shows that if the economy is growing to the steady state, wealth inequality increases monotonically along the transition whenever minimum consumption requirements are not zero.

We understand that this paper extends and complements the existing related lit-
erature mentioned above in some nontrivial dimensions. It is the natural framework
to directly compare our results with the literature studying asset trading volume and
thus to investigate the implications of capital accumulation on the evolution of equi-
librium portfolios. It shows that the nonstationary distribution of aggregate wealth
studied by Bossaerts and Zame [2005]) can be the natural consequence of capital accu-
mulation, even in simple environments like ours. Moreover, we provide relatively sim-
ple algorithms to compute equilibrium portfolios in the standard neoclassical growth
model under alternative complete market structures. Since this framework was the
natural benchmark to study quantitatively the behavior of asset prices, it can also be
used to evaluate quantitatively its predictions about the evolution of asset holdings,
stock trading volume, etc.

The paper is organized as follows. In Section 2, we describe the economy and
characterize the set of Pareto optimal allocations. In Section 3, we study equilibrium
portfolios in the corresponding economy with sequential trading and dynamically
complete markets. Section 4 concludes. All proofs are in the Appendix.

2 The Economy

We consider an economy populated by $I$ (types of) infinitely-lived agents where $i \in$
{$1, \ldots, I$}. Time is discrete and denoted by $t = 0, 1, 2, \ldots$. Each agent is endowed with
one unit of time every period but for simplicity we assume that leisure is not valued.
There is only one consumption good which can be either consumed or invested.
Goods invested transform one-to-one in new capital next period. There is a constant
returns to scale aggregate technology to produce the consumption good operated
by a representative firm. This technology is subject to productivity shocks and $s_t$
represents the realization of this shock at date $t$. We assume that $\{s_t\}$ is a finite
state first-order stationary Markov process. Transition probabilities are denoted by
\( \pi(s, s') \) where \( s_t, s, s' \in S = \{s_1, ..., s_N\} \). Let \( s' = (s_0, ..., s_t) \in S^{t+1} \) represent the partial history of aggregate shocks up to date \( t \). We write \( s^{t+1}/s^t \) to denote that \( s^{t+1} \) is an immediate successor of \( s^t \). These histories are observed by all the agents. The probability of \( s^t \) is constructed from \( \pi \) in the standard way and \( x(s^t) \) denotes the value of \( x \) at the node \( s^t \).

A consumption bundle for agent \( i \) is a sequence of functions \( \{c_t\}_{t=0}^{\infty} \) such that \( c_t : S^{t+1} \rightarrow [\gamma_i, +\infty) \) for all \( t \) and \( \sup_{s^t} c(s^t) < \infty \). \( C_i \) is agent \( i \)'s consumption set with all these sequences as elements. The parameter \( \gamma_i \geq 0 \) is interpreted as the minimum consumption required by agent \( i \). Agent \( i \)'s preferences on \( C_i \) are represented by expected, time-separable, discounted utility. That is, if \( c_t \in C_i \) then:

\[
U(c_t) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u_i(c_t(s^t)),
\]

where \( \beta \in (0, 1) \) and the momentary utility function \( u_i \) belongs to the following class:

\[
u_i(c) = \begin{cases} 
\frac{(c-\gamma_i)^{1-\sigma}}{1-\sigma} & \text{if } \sigma \neq 1 \text{ and } \sigma < 1 \\
\ln(c-\gamma_i) & \text{if } \sigma = 1 
\end{cases}
\]

where \( (c-\gamma_i) \geq 0 \).

Let \( K(s^t) \geq 0 \) denote the stock of capital chosen by the representative firm at the node \( s^t \) and available in period \( t + 1 \) to produce with the aggregate technology. The depreciation rate is given by \( \delta \in (0, 1) \). Let \( sF(K, L) \) represent this technology, where \( K \) is the stock of capital available, \( L \) is the level of labor and \( s \) is the productivity shock. \( F : \mathbb{R}_{+}^{2} \rightarrow \mathbb{R}_{+} \) is homogeneous of degree 1, concave, strictly increasing, continuously differentiable and satisfies for all \( L > 0 \) : (a) \( \partial F(0, L)/\partial K = \infty \) and (b) \( \lim_{K \rightarrow \infty} \partial F(K, L)/\partial K = 0 \). Condition (a) rules out corner solutions. More importantly, condition (b) guaranties that there exists some \((\underline{K}, \overline{K})\) such that \( 0 \leq \underline{K} \leq K(s^t) \leq \overline{K} \) for all \( s^t \) since consumption must be bounded from below.\(^2\) Without loss of generality, we can restrict ourselves to \( K(s^t) \in X \equiv [\underline{K}, \overline{K}] \)

\(^2\)More specifically, \( \overline{K} \) solves \( \min_{s_1, \delta} \sum_{t=0}^{\infty} \gamma_i \) and \( \overline{K} \) is the standard upper
for all $s^t$. We denote $K = (K(s^t))_{s^t}$ where $K_0 \in X$ is the initial stock of capital which is assumed to be greater than $K$. Note that $L(s^t) = I$ for all $s^t$ and denote $f(K(s^{t-1})) = F(K(s^{t-1}), I)$. Below we will refer to the special case of unproductive labor when $F(K, L) = F(K)$ for all $L$ and consequently $F_L = 0$ for all $(K, L)$.

To compute equilibrium portfolios we proceed as follows. We first characterize the set of Pareto optimal allocations. Under our assumptions about preferences, the problem reduces to solve for aggregate variables given the existence of a fictitious aggregate representative consumer (ARC). Then, in the next section we proceed studying a competitive market arrangement with sequential trading to analyze the evolution of equilibrium portfolios.

\textit{Planner’s Problem}

Under our concavity assumptions on both utility and the production functions, the set of Pareto optimal allocations can be parametrized by welfare weights. Suppose that $\alpha_i$ is the welfare weight assigned by the planner to agent $i$. Given a vector of welfare weights $\alpha = (\alpha_i)_{i=1}^I$, the planner’s problem is given by:

$$V(s_0, K_0; \alpha) = \max \sum_{i=1}^I \alpha_i \left\{ \sum_{t=0}^\infty \sum_{s^t} \beta^t \pi(s^t) \left( c_i(s^t) - \frac{\gamma_i^{1-\sigma}}{1-\sigma} \right) \right\},$$

(PP)

subject to

$$\sum_{i \in Y} c_i(s^t) + K(s^t) - (1 - \delta)K(s^{t-1}) = s_t f(K(s^{t-1})) \quad \text{for all } s^t,$$

$$c_i \in C_i, \ K(s^t) \in X \quad \text{for all } s^t \text{ and all } i,$$

$$K_0 \in X \text{ given}.$$

Under these preference representations, it is well-known that the solution to (PP) is equivalent to solve:

\textit{bound in the neoclassical growth model.} 

\textit{That is, the utility possibility set is convex and closed and thus we can immediately apply the supporting hyperplane theorem.}
\[ V(s_0, K_0) = \max_{(c, K)} \sum_{t=0}^{\infty} \sum_{s_t} \beta^t \pi(s_t) \frac{(C(s_t) - \gamma)^{1-\sigma}}{1 - \sigma}, \] 

subject to

\[ C(s_t^t) + K(s_t^t) - (1 - \delta)K(s_{t-1}^t) = s_t f(K(s_{t-1}^t)) \quad \text{for all } s_t, \]

\[ C(s_t^t) - \gamma \geq 0, \quad K(s_t^t) \in X \quad \text{for all } s_t, \]

\[ K^0 \in X \text{ given,} \]

where \( \gamma = \sum_{i=1}^{I} \gamma_i \) and \( C(s_t^t) \) is aggregate consumption. Given a vector \( \alpha \), optimal individual consumption for each \( i \) is determined by:

\[ [c_i(s_t^t; \alpha) - \gamma_i] = \frac{(\alpha_i)^{1/\sigma}}{\sum_{h=1}^{I} (\alpha_h)^{1/\sigma}} [C(s_t^t) - \gamma]. \quad (1) \]

The equivalence can be easily determined because (1) satisfies the necessary and sufficient first order conditions for the problem (\( PP \)) and the solution to (\( APP \)) is immune to affine linear transformations of preferences.\(^4\) Very importantly, this implies that the evolution of the stock of capital is independent of the initial distribution of wealth. Note that (1) implies that the marginal rate of substitution of any agent equals the ARC’s marginal rate of substitution.

The former discussion reduces the problem to the traditional one-sector neoclassical growth model under uncertainty. It is a standard exercise to establish that this problem has a recursive formulation.\(^5\) The value function \( V : S \times X \to \mathbb{R} \) solves the following functional equation:

\[ V(s, K) = \max_{(C,K')} \left\{ \frac{(C - \gamma)^{1-\sigma}}{1 - \sigma} + \beta \sum_{s'} \pi(s,s') V(s', K') \right\}, \quad (\text{RAPP}) \]

\(^4\)Here \( \sum_{i=1}^{I} \alpha_i (\alpha_i)^{1/\sigma} \) reduces to \( (C(s_t^t) - \gamma)^{1-\sigma} \frac{1}{1 - \sigma} \left( \sum_{i=1}^{I} (\alpha_i)^{1/\sigma} \right)^{\sigma} \) for each \( s_t^t \).

\(^5\)See Stokey, Lucas and Prescott [1989, Section 10.1]). If \( \sigma \geq 1 \), utility functions are unbounded from below. To deal with this case, the techniques explained in Alvarez and Stokey [1998] can be easily adapted. For that, notice that momentary utility functions are homogeneous in \( c - \gamma_i \) and the problem can be transformed to the optimal growth model with a production function \( sf(k) - \sum \gamma_i \).
subject to

\[ C + K' - (1 - \delta)K = sf(K), \]

\[ C - \gamma \geq 0, \quad K' \in X. \]

Moreover, it can also be shown that \( V \) is strictly increasing, strictly concave in \( K \) and continuously differentiable in the interior of \( X \). The solution to the problem (RAPP) is a set of continuous policy functions \((C(s, K), K'(s, K))\). To fully characterize the set of Pareto optimal allocations, we construct the recursive version of individual consumption (1) by:

\[
[c_i(s, K, \alpha) - \gamma_i] = \frac{(\alpha_i)^{1/\sigma}}{\sum_{j=1}^{I} (\alpha_j)^{1/\sigma}} [C(s, K) - \gamma], \tag{2}
\]

for all \((s, K)\), given a vector of welfare weights \(\alpha\).

Next we will consider competitive decentralization with sequential trading and complete markets to study asset trading and its corresponding equilibrium portfolios.

### 3 Sequential Trading and Recursive Competitive Equilibrium

We assume the following trading opportunities. Every period \( t \) and after having observed \( s^t \), agents meet in spot markets to trade the consumption good and different assets. They can trade firm’s shares, where \( \theta_i(s^t) \) is the number of shares chosen by agent \( i \) at \( s^t \). There is one outstanding share and then \( \sum_i \theta_i(s^t) = 1 \) for all \( s^t \).\(^6\) We do not impose short-selling constraints. We assume that agent \( i \) is endowed with \( \theta_i(s_{-1}) = \theta_i^0 > 0 \) shares of the firm at date 0. It is assumed that for each agent \( i \) this share is large enough such that the optimal consumption path is in the interior of the

\(^6\) Firm’s financial policies do not affect equilibrium allocations and prices since the Modigliani-Miller theorem holds in this framework.
consumption set (see the Appendix for details). Let \( p(s^t) \) be the ex-dividend price of one share at \( s^t \). Let \( w(s^t) \) be the wage paid by the firm per unit of labor at \( s^t \).

Agents can also trade a complete set of fully enforceable Arrow securities with zero net supply. The Arrow security \( s' \) traded at \( s^t \) pays one unit of consumption next period if \( s^{t+1} = (s^t, s') \) and 0 otherwise. Let \( q(s^t)(s') \) be the price of this security at \( s^t \).

Denote \( a_i(s^t, s') \) as agent \( i \)'s holdings of this security. To rule out Ponzi schemes, we restrict agents to bounded trading strategies. These (implicit) bounds are assumed to be sufficiently large such that they do not bind in equilibrium. All these prices are in units of the \( s^t \)-consumption good.

The firm chooses labor and accumulates capital to maximize its value. In this framework with sequential trading, if the firm reaches \( s^t \) with a stock of capital \( K(s^{t-1}) \) then its problem can be expressed:

\[
V_F(s^t, K(s^{t-1})) = \max_{K(s'), L(s')} \left\{ d(s^t, K(s^{t-1})) + \sum_{s'} q(s^t)(s') V_F(s^t, s', K(s')) \right\}, \quad \text{(FP)}
\]

subject to

\[
d(s^t, K(s^{t-1})) = s_t F(K(s^{t-1}), L(s^t)) + (1 - \delta) K(s^{t-1}) - K(s^t) - w(s^t)L(s^t), \quad \text{(3)}
\]

for all \( s^t \). Here, \( V_F(s^t, K(s^{t-1})) \) is the value of the firm with stock of capital \( K(s^{t-1}) \) at \( s^t \).

From the consumer side, given a price system \( (p, q, w) \) agent \( i \)'s problem is given by:

\[
\max_{(c_i, a_i, \theta_i)} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) \frac{(c_i(s^t) - \gamma_i)^{1-\sigma}}{1 - \sigma}, \quad \text{(AP)}
\]

subject to

\[
c_i(s^t) + p(s^t)\theta_i(s^t) + \sum_{s'} q(s^t)(s') a_i(s^t, s') = \phi_i(s^t) + w(s^t), \quad \text{(4)}
\]

\[
\phi_i(s^t, s') = [p(s^t, s') + d(s^t, s')] \theta_i(s^t) + a_i(s^t, s'), \quad \text{(5)}
\]
where \( c_i \in C_i \) and \((a_i, \theta_i)\) are bounded. Here, \( \phi_i(s^t) \) denotes agent \( i \)'s non-human wealth at \( s^t \). A competitive equilibrium in this framework is defined in the standard way.

**Definition 1** A *Competitive Equilibrium with Sequential Trading (STCE)* is a price system \((\hat{p}, \hat{q}, \hat{w})\), an allocation \( \{(\hat{c}_i)_i, \hat{K}, \hat{L}\} \) and equilibrium portfolios \((\hat{a}_i, \hat{\theta}_i)_i\) such that:

**(STCE 1)** Given \((\hat{p}, \hat{q}, \hat{w})\), \((\hat{c}_i, \hat{\theta}_i, \hat{a}_i)\) solves agent \( i \)'s problem \((AP)\) for each \( i \).

**(STCE 2)** Given \((\hat{p}, \hat{q}, \hat{w})\), \((\hat{K}, \hat{L})\) solves the firm’s problem \((FP)\) where \( \hat{L}(s^t) = I \) for all \( s^t \).

**(STCE 3)** All markets clear. For all \( s^t : 
\begin{align*}
\sum_{i=1}^{I} c_i(s^t) + [K(s^t) - (1 - \delta)K(s^{t-1})] & = s_t f(K(s^{t-1})), \\
\sum_{i=1}^{I} \hat{a}_i(s^t, s') & = 0 \text{ for all } s', \\
\sum_{i=1}^{I} \hat{\theta}_i(s^t) & = 1.
\end{align*}

With these preferences representations, we can establish the linearity of individual consumption with respect to his total wealth. Let \( P(s^{t+n}/s^t) \) be the price of one unit of the consumption good to be delivered at \( s^{t+n} \) in terms of the \( s^t \)-consumption good (where \( P(s^t/s^t) = 1 \)). We define \( V_P(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}/s^t} P(s^{t+n}/s^t) \) and is interpreted as the present discounted value of a perpetual bond that pays 1 unit of the consumption good in each period and in each state. Let \( M(s^{t+n}/s^t) = \frac{P(s^{t+n}/s^t)}{\beta^{t+n} \pi(s^{t+n})} \) and define \( V_X(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}/s^t} P(s^{t+n}/s^t) \left( M(s^{t+n}/s^t) / \beta^{t+n} \pi(s^t) \right) \). \( V_X \) is interpreted as the presented discounted value of aggregate consumption growth exceeding minimum requirements. Finally, the value of human wealth at \( s^t \) is given by \( V_w(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}/s^t} P(s^{t+n}/s^t) w(s^{t+n}/s^t) \), which is the same for all agents.
Lemma 2 Agent i’s consumption can be expressed by:

\[ c_i(s^t) = \gamma_i \left( 1 - [VX(s^t)]^{-1} \phi_i(s^t) \right) + [VX(s^t)]^{-1} \left( \phi_i(s^t) + V_w(s^t) \right), \]

for all \( s^t \) and all \( i \).

The affine linearity of consumption with respect to total wealth will be an important aspect to simplify the analysis of our main result. This is a direct consequence of the particular preference representation we have purposely assumed.

The Markovian structure of this economy assures that there exists an equivalent Recursive Competitive Equilibrium (RCE) for each STCE.\(^7\) Consider the set of state variables. At the consumer level, it is described by individual wealth, \( \phi_i \). At the firm level, \( k \) describes the firm’s stock of capital. Let \( \Phi \) and \( K \) describe the distribution of wealth and the aggregate stock of capital, respectively. Therefore, the set of aggregate state variables at the aggregate level is fully described by \( (s, \Phi, K) \). The price system is given by \( p, Q, w : S \times \mathbb{R}_+^I \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) representing prices for shares, Arrow securities and wages, respectively. To simplify notation, we directly impose the labor market equilibrium condition such that \( L(s, \Phi, K) = I \) for all \( (s, \Phi, K) \).

Definition 3 A RCE is a set of value functions for the individuals \( (V_i) \), a value function for the firm \( V_F \), a set of policy functions for the individuals \( (c_i, a_i^t, \theta_i^t) \), a policy function for the firm \( (k^t) \), a set of prices \( (p, Q, w) \) and laws of motion for the aggregate state variables \( \Phi^t = G(s, \Phi, K) \) and \( K^t = H(s, \Phi, K) \) such that:

(RCE 1) Given \((p, Q, w)\), for each agent \( i \) \((c_i, a_i^t, \theta_i^t)\) are the corresponding policy functions and \((V_i, c_i, a_i^t, \theta_i^t)\) solves:

\[ V_i(\phi_i, s, \Phi, K) = \sup_{(c_i, a_i^t, \theta_i^t)} \left\{ u(c_i) + \beta E \left[ V_i(\phi_i^t, s^t, \Phi^t, K^t) \mid s \right] \right\}, \]

\(^7\)See Ljungqvist and Sargent [2004, Chapter 8 and 12] for details.
subject to

\[ c_i + p(s, \Phi, K)\theta'_i + \sum_{s'} Q(s, \Phi, K)(s')a'_i(s') = \phi_i(s, \Phi, K) + w(s, \Phi, K), \]

\[ \phi'_i(s', \Phi', K') = [p(s', \Phi', K') + d(s', \Phi', K')] \theta'_i + a'_i(s'), \]

where \( \Phi' = G(s, \Phi, K) \) and \( K' = H(s, \Phi, K) \).

**(RCE 2)** Given \((p, Q, w)\), \(V_F\) is the recursive version solving \((FP)\) and \((k')\) solves the firm’s problem, where dividends are given by:

\[ d(s, k, \Phi, K) = sf(k) + (1 - \delta)k - k'(k, s, \Phi, K) - w(s, \Phi, K)I. \]

**(RCE 3)** All markets clear. For all \((\phi_i, k, s, \Phi, K)\):

\[ \sum_i c_i(\phi_i, s, \Phi, K) + (k'(k, s, \Phi, K) - (1 - \delta)k) = sf(k), \]

\[ \sum_i a'_i(\phi_i, s, \Phi, K)(s') = 0 \quad \text{for all } s', \]

\[ \sum_i \theta'_i(\phi_i, s, \Phi, K) = 1. \]

**(RCE 4)** Consistency. For all \((s, \Phi, K)\) and each \(i\):

\[ K' = H(s, \Phi, K) = k'(s, K, \Phi, K), \]

\[ \Phi'_i = G_i(s, \Phi, K) = [p(G(s, \Phi, K), H(s, \Phi, K)) + d(G(s, \Phi, K), H(s, \Phi, K))]\theta'_i(\Phi_i, s, \Phi, K) + a'_i(\Phi_i, s, \Phi, K). \]

We are interested in decentralizing a particular Pareto optimal allocation as a RCE defined above. With that purpose, consider the policy functions \((C, K')\) for the problem \((RPP)\). Define the stochastic discount factor as follows:

\[ Q(s, K)(s') = \beta \pi(s, s')(C(s', K'(s, K)) - \gamma)^{-\sigma}, \quad (7) \]
for all \((s, K, s')\). Individual consumption is constructed using (2). It will be useful to have a recursive version of (6). The value of human wealth, which is independent of welfare weights \(\alpha\), can be expressed as follows:

\[
V_w(s, K) = w(s, K) + \sum_{s'} Q(s, K)(s')V_w(s', K'(s, K)).
\]  

(8)

Also, define:

\[
V_P(s, K) = 1 + \sum_{s'} Q(s, K)(s')V_P(s', K'(s, K)),
\]

\[
V_X(s, K, s') = \frac{C(s', K'(s, K)) - \gamma}{C(s, K) - \gamma} + \sum_{z} Q(s', K'(s, K))(z)V_X(s', K'(s, K), z),
\]

where \(V_X(s, K) = \sum_{s'} Q(s, K)(s')V_X(s, K, s')\). It can be shown that each of the previous functional equations have a unique continuous solution \(V_w\), \(V_P\) and \(V_X\). See the Appendix for technical details.

Individual consumption must thus satisfy the recursive version of (6):

\[
c_i(s, K, \alpha) = \gamma_i (1 - b(s, K)) + z(s, K) (\phi_i(s, K, \alpha) + V_w(s, K)),
\]

(9)

where \(b(s, K) = [1 + V_X(s, K)]^{-1} V_P(s, K)\) and \(z(s, K) = [1 + V_X(s, K)]^{-1}\).

Note that (2) and (9) imply that the ARC’s consumption is given by:

\[
C(s, K) = \gamma (1 - b(s, K)) + z(s, K) (V_P(s, K, \alpha) + V_w(s, K)I),
\]

which is precisely the expression expected to be obtained under these preference representations for the ARC.

We are ready now to decentralize a Pareto optimal allocation as a RCE with zero initial transfers, conditional upon our assumptions about the initial distribution of shares \(\theta^0 = (\theta^0_i)_{i=1}^I\). Let \((s_0, K_0)\) be the date 0 state of the economy. We will follow Negishi’s [1960] approach and show that given \((s_0, K_0, \theta^0)\) there exists a vector \(\alpha^0 = \alpha(s_0, K_0, \theta^0)\) such that the corresponding Pareto optimal allocation can be
decentralized as a RCE with zero initial transfers for each agent. From now on, we impose the consistency conditions (RCE 4) and thus we avoid writing policy functions depending on individual state variables.

Note first that we allow for trading in both Arrow securities and shares. This means that there are redundant assets since we have \( S + 1 \) assets with linearly independent payoffs while there are \( S \) states of nature. To illustrate our results, we shut down stock trading where we impose \( \theta_i'(s, \Phi, K) = \theta_i^0 \) for all \( i \) and all \( (s, \Phi, K) \). As discussed below, it is a relatively simple exercise to apply the analytical tools developed here to study and explicitly determine asset trading and equilibrium portfolios in this framework with arbitrary complete market structures.

Define wages and stock prices as follows:

\[
w(s, K) = sF_2(K, I), \tag{10}
\]

\[
p(s, K) = \sum_{s'} Q(s, K)(s')[p(s', K'(s, K)) + d(s', K'(s, K))]. \tag{11}
\]

To determine individual wealth, consider the following functional equation constructed from the agent \( i \)'s budget constraint (18):

\[
\phi_i(s, K, \alpha) = c_i(s, K, \alpha) - w(s, K) + \sum_{s'} Q(s, K)(s')\phi_i(s', K'(s, K), \alpha). \tag{12}
\]

As mentioned before, both the operator determining the stock price (11) and individual wealth (12) will have unique continuous functions as solutions. Extending Espino and Hintermaier [2005], we show below that there exists a unique \( \alpha^0 = \alpha(s_0, K_0, \theta^0) \) such that:

\[
\phi_i(s_0, K_0, \alpha^0) = \theta_i^0 [p(s_0, K_0) + d(s_0, K_0)] = \theta_i^0 V_F(s_0, K_0).
\]

Consequently, the Pareto optimal allocation corresponding to \( \alpha^0 \) can be decentralized as a RCE with zero initial transfers.
We fix $\alpha^0$ to obtain equilibrium Arrow security holdings as follows. First, define:

$$A_i(s, K, \alpha^0) = \phi_i(s, K, \alpha^0) - \theta^0_i \left[ p(s, K) + d(s, K) \right].$$

Therefore, equilibrium portfolios in this RCE will be given by:

$$\theta'_i(s, \Phi, K) = \theta^0_i,$$  \hspace{1cm} (13)

$$a'_i(s, \Phi, K)(s') = A_i(s', K'(s, K), \alpha^0),$$

where $\Phi_i = \phi_i(s, K, \alpha^0)$ for each $i$ is uniquely determined by $(s, K)$ given $\alpha^0$.

We can now summarize this discussion with the following result.

**Proposition 4** There exists a unique welfare weight $\alpha^0 = \alpha(s_0, K_0, \theta^0)$ such that the corresponding Pareto optimal allocation can be decentralized as a RCE. The price system is given by (7), (11) and (10), individual consumption by (2) and equilibrium portfolios by (13).

In the Appendix we show how to explicitly express $\alpha^0$ and $A_i$’s as functions of $V_F, V_w, V_p, \theta^0$ and $\gamma_i$’s.

We say that the *fixed equilibrium portfolio property* is satisfied if for each $s'$, for all $(s, \Phi, K)$ and for all $i$, it follows that $A'_i(s, \Phi, K)(s') = A'_i(s')$. That is, individual portfolios are independent of the aggregate state of the economy. The next result shows that if this framework reduces to an endowment economy with unproductive capital, then this property holds. Whenever capital is unproductive and $\delta = 0$, we get a version of the stationary Lucas tree model with heterogeneous agents which is a particular version of JKS [2003].

**Proposition 5** (JKS [2003]) In the endowment version of this economy, equilibrium portfolios are fixed such that:

$$a'_i(s, \Phi)(s') = a'_i(s')$$  \hspace{1cm} for all $(s, \Phi)$ and for all $s'$,
where $\Phi_i = \phi_i(s, \alpha^0)$ for each $i$.

It is important to notice that the demand for Arrow securities is independent of the unique aggregate state variable, $s$. That is, independently of the state $s$ today (and thus independent of the uniquely determined wealth distribution, $\Phi(s)$), agents construct a fixed equilibrium portfolio of Arrow securities.

It turns out that if minimum consumption requirements and labor income are both assumed away, this environment is still unable to generate changing equilibrium portfolios and consequently the fixed equilibrium portfolio property is preserved.

**Proposition 6** If $\gamma_i = 0$ for all $i$ and $w(s, K) = 0$ for all $(s, K)$, then the equilibrium portfolio is kept fixed and given by:

$$\theta_i'(s, \Phi, K) = \theta_i^0,$$

$$d_i'(s, \Phi, K)(s') = 0, \text{ for all } s' \text{ and all } (s, \Phi, K).$$

Now we show that this result might not be robust in the more general framework introduced above. First, if minimum consumption requirements differ from 0 for at least one agent, then the trading strategy of fixed portfolios cannot be optimal in equilibrium even if there is no labor income.

**Proposition 7** Suppose that labor is unproductive with $w(s, K) = 0$ for all $(s, K)$. If $\gamma_i > 0$ for some agent $i$, then fixed portfolios cannot be optimal in equilibrium.

Secondly, we show that under certain conditions the introduction of labor income (productive labor) is sufficient to generate changing equilibrium portfolios.

**Proposition 8** Assume away minimum consumption requirements where $\gamma_i = 0$ for all $i$. If $w(s, K) > 0$ for all $(s, K)$ and human wealth is linearly independent of non-human wealth, then fixed portfolios cannot be optimal in equilibrium.
It is important to notice that under the assumptions of Proposition 7, the fixed portfolio trading strategy will not be optimal in equilibrium even if the initial heterogeneity in shares is assumed away (e.g., $\theta_i^0 = 1/I$ for all $i$) whenever $\gamma_i \neq \gamma_h$ for some $i, h$. On the other hand, if this is the case under the assumptions of Proposition 8, agents are ex-ante identical and consequently there is no asset trading in equilibrium.

Discussion of The Main Result

To simplify the discussion, suppose that $\gamma_i = \bar{\gamma}$ for all $i$. Hence, at any state $(s, K)$ the degree of heterogeneity across agents is represented by their relative participation in aggregate non-human wealth. Whenever this participation changes with the current state $(s, K)$, the associated equilibrium portfolios will in general adjust accordingly.

To see this more concretely, consider the evolution of individual non-human wealth. A recursive version of (19) in the Appendix implies that:

$$\phi_i(s', K'(s, K)) = \frac{\gamma}{\phi_i(s, K)} [V_p(s', K'(s, K)) - V_p(s, K)\Gamma(s, K, s')]$$

$$+ \Gamma(s, K, s')$$

$$+ \frac{1}{\phi_i(s, K)} [V_w(s, K)\Gamma(s, K, s') - V_w(s', K'(s, K))]$$

where $\Gamma(s, K, s') = \frac{z(s, K)}{z(s', K'(s, K))} \left( \frac{\beta_{\pi(s, s')}}{Q(s, K)(s')} \right)^{1/\sigma}$.

Agent $i$'s stochastic growth of wealth in general depends on his individual level of wealth. This holds except in the case where minimum consumption requirements are 0 for all agents and labor is unproductive. In that particular case, the fraction of aggregate wealth for each agent is kept constant at $\theta_i^0$. At any aggregate state $(s, K)$, individual wealth will be perfectly correlated with aggregate wealth. Thus, individual risk coincides with aggregate risk and consequently there is no trade in equilibrium.

When minimum consumption requirements are 0 but there is labor income (e.g. $w(s, K) > 0$ for all $(s, K)$), it is interesting to notice that each agent $i$'s partici-
participation of in aggregate total wealth, given by $V_F(s, K) + V_w(s, K)I$, is kept constant as time and uncertainty unfold. Equilibrium portfolios must then adjust such that this property holds if $V_F(s, K)$ and $V_w(s, K)$ are not linearly related for all $(s, K)$. The dependence with respect to $(s, K)$ comes from the construction of asset holdings, $a_i'(s, K, A)(s') = A_i(s', K'(s, K))$ where $A$ is uniquely determined by $A_i(s, K)$. Remember that $\phi_i(s', K'(s, K)) = \theta_i V_F(s', K'(s, K)) + A_i(s', K'(s, K))$ and thus the equilibrium portfolio choice depends upon the state $(s, K)$ because this determines the stock of capital next period. If $V_F(s, K)$ and $V_w(s, K)$ are not linearly independent, the distribution of consumption is independent of the initial aggregate state $(s_0, K_0)$. When both sources of wealth are linearly dependent, the evolution of the aggregate stock of capital will have no impact on the individual composition of wealth in equilibrium. Consequently, the equilibrium level of trading is reduced to zero. For example, in the particular case with log preferences, full depreciation of capital and a Cobb-Douglas production function, it is possible to show that $V_F(s, K)$ and $V_w(s, K)$ are constant fractions of aggregate output. This implies that the corresponding vector of welfare weights is independent of $(s, K)$ and therefore $A_i(s', K'(s, K)) = 0$ for all $i$, for all $(s, K)$ and all $s'$. It is well-known, however, that this linear dependence is not expected to be obtained in general since this is a very particular property of that particular example.

With unproductive labor but nonzero minimum consumption requirements, the intuition is similar. In that case, these $\gamma_i$'s work as "negative endowment" for each agent and thus its value is one of the determinants to explain the evolution of individual wealth relative to the aggregate level. Again, note that the participation of each agent $i$ in aggregate total wealth, now given by $V_F(s, K) - \gamma V_F(s, K)$, is kept constant in this case as well. Thus, equilibrium portfolios will adjust conditional
upon \((s, K)\) such that this property holds.

4 Conclusions

This paper studies equilibrium portfolios in the traditional one-sector neoclassical growth model under uncertainty with heterogeneous agents. Preferences are purposely restricted such that Engel curves are affine linear in lifetime human and non-human wealth. Heterogeneity across agents can arise due to differences in initial non-human wealth and minimum consumption requirements.

JKS [2003] have seriously questioned the ability of the stationary Lucas tree model to generate nontrivial asset trading under alternative complete market structures and fairly general patterns of heterogeneity across agents. They show that agents choose a fixed equilibrium portfolio which is independent of the state of nature. They conclude that some friction (informational, financial, etc.) must play a significant role in generating nontrivial asset trading in that framework.

In this paper, a crucial aspect initially pointed out in Espino and Hintermaier [2005] have been isolated and studied in more detail. That is: whenever the environment under study generates changing degrees of heterogeneity across agents, the trading strategy of fixed portfolios cannot be optimal in equilibrium.

This paper purposely abstracts from all kind of frictions. The analysis shows how to explicitly determine equilibrium portfolios as a function of different sources of income, preferences and the initial endowment of wealth. We have shown that our environment can generate changing heterogeneity either if minimum consumption requirements are not zero or if labor income is not zero and the value of human wealth and non-human are linearly independent.

The theoretical framework analyzed here can be easily extended to study more general complete market structures as, for example, those discussed in detail by
Espino and Hintermaier [2005]. It can also be adapted to deal with accumulation technologies allowing for adjustment costs as in Jermann [1998] and Boldrin, Christiano and Fisher [2001]. The introduction of adjustment costs was important to study quantitatively asset returns in production economies. Very importantly, we understand that the framework presented here is the natural benchmark to test through quantitative experiments the predictions of the celebrated neoclassical growth model in terms of the evolution of asset holding and stock trading volume. We do not claim that these crucial quantitative aspects will be fully explained by the simple framework presented here. Some other candidates to generate additional trading might be required. However, we do consider important to disentangle the specific contribution of each of those alternative sources and thus the environment described here seems a natural first step.

Appendix

Proof of Lemma 2. Given our assumptions, we can fully characterize an interior solution for the agent’s problem.\(^8\) Let \(\lambda_i(s^t)\) be the Lagrange multiplier corresponding to agent \(i\)’s budget constraint at \(s^t\). Agent \(i\)’s necessary and sufficient first order conditions imply that:

\[
\beta^t \pi(s^t) \left( c_i(s^t) - \gamma_i \right)^{-\sigma} = \lambda_i(s^t), \tag{15}
\]

\[
\lambda_i(s^t, s') = q(s^t)(s') \lambda_i(s^t), \tag{16}
\]

\[
p(s^t) = \sum_{s'} q(s^t)(s') \left[ p(s^t, s') + d(s^t, s') \right]. \tag{17}
\]

\(^8\)It is well-known that the necessary and sufficient transversality condition will hold in this framework.
for all \( s^t \) and all \( s' \). Note (17) coupled with (5) implies that the budget constraint can be rewritten as follows:

\[
c_i(s^t) + \sum_{s'} q(s')(s') \phi_i(s', s') = \phi_i(s^t) + w(s')
\]

(18)

for all \( s' \). Given the definitions of \( P(s^{t+n}/s^t) \), \( M(s^{t+n}/s^t) \), \( V_X(s^t) \), \( V_P(s^t) \) and \( V_w(s^t) \), note that (15) can be rewritten as follows:

\[
c_i(s^{t+n}) - \gamma_i = \lambda_i(s^t) \frac{1}{\sigma} \left( \frac{P(s^{t+n}/s^t)}{\beta^{t+n}(s^{t+n}/s^t)} \right)^{1/\sigma},
\]

for all \( s^{t+n} \). Using this expression and the necessary transversality condition in (18), we can repeatedly replace non-human wealth to get:

\[
(\lambda_i(s^t))^{1/\sigma} M(s^t)^{1/\sigma} \sum_{n=0}^{\infty} \sum_{s^{t+n}/s^t} P(s^{t+n}/s^t) \left( \frac{M(s^{t+n}/s^t)}{M(s^t)} \right)^{1/\sigma} = \phi_i(s^t) + V_w(s^t) - \gamma_i \sum_{n=0}^{\infty} \sum_{s^{t+n}/s^t} P(s^{t+n}/s^t),
\]

Individual consumption can then be expressed by:

\[
c_i(s^t) = \gamma_i \left( 1 - [V_X(s^t)]^{-1} V_P(s^t) \right) + [V_X(s^t)]^{-1} (\phi_i(s^t) + V_w(s^t)),
\]

as in (6). To determine \( \phi_i(s^t, s') \) (and thus the evolution of individual wealth), observe that:

\[
(c_i(s^t, s') - \gamma_i) = \left( \frac{q(s^t)(s')}{\beta \pi(s_t, s')} \right)^{-1/\sigma} (c_i(s^t) - \gamma_i),
\]

and thus (6) implies that for all \( (s^t, s') \):

\[
\begin{align*}
\phi_i(s^t, s') &= \gamma_i (V_P(s^t, s') - \Gamma(s^t, s') (V_P(s^t, s'))) + \Gamma(s^t, s') \phi_i(s^t) \\
&= \Gamma(s^t, s') \phi_i(s^t) - (V_w(s^t, s') - \Gamma(s^t, s') V_w(s^t)),
\end{align*}
\]

where \( \Gamma(s^t, s') = \frac{[V_X(s^t)]^{-1} \beta \pi(s_t, s')}{[V_X(s^t, s')]^{-1} q(s^t)(s')}^{1/\sigma} \).
Let $\mathcal{C}(S \times X)$ be the set of continuous mapping $S \times X$ into the real numbers. For any continuous function $r : S \times X \to \mathbb{R}$, consider the operator $T$ defined by:

$$(TR)(s, K) = r(s, K) + \sum_{s'} Q(s, K)(s') R(s', K'(s, K)).$$

**Lemma 9** Suppose that $r(s, K)$ is one of the following functions:

(a) $C(s, K)$, (b) $w(s, K)$, (c) $d(s, K)$, (d) a constant, (e) $C(s, K)$.

Then, in all these cases there exists a unique continuous function $R \in \mathcal{C}(S \times X)$ such that $R(s, K) = (TR)(s, K)$ for all $(s, K)$.

**Proof of Lemma 9.** Since the value function $V$ in (RAPP) is strictly increasing, strictly concave and differentiable in the interior of $X$ and the corresponding policy functions are continuous, we can proceed as in Espino and Hintermaier [2005]. Technical details are available upon request. ■

**Proof of Proposition 3.** We normalize $\sum_{j=1}^{I} (\alpha_j)^{1/\sigma} = 1$. Let $(C(s, K), K'(s, K))$ be the set of continuous policy functions solving the problem (RAPP). Compute first the value of aggregate consumption as the solution of the following functional equation:

$$V_C(s, K) = C(s, K) + \sum_{s'} Q(s, K)(s') V_C(s', K'(s, K)).$$

Note that:

$$V_F(s, K) = d(s, K) + \sum_{s'} Q(s, K)(s') V_F(s', K'(s, K)),
$$

is independent of $\alpha$. Since $w(s, K) = sF_2(K, I)$, we have that:

$$V_w(s, K) = w(s, K) + \sum_{s'} Q(s, K)(s') V_w(s', K'(s, K)),
$$

is independent of $\alpha$ as well. Note that $C(s, K) = d(s, K) + w(s, K)I$ and thus $V_C(s, K) = V_F(s, K) + V_w(s, K)I$ for all $(s, K)$. Lemma 9 implies that there exist unique continuous functions $V_C, V_F$ and $V_w$. 

23
The value of individual consumption corresponding to the Pareto optimal allocation \( \alpha \) is given by:

\[
V^i_C(s, K, \alpha) = \hat{c}^i(s, K, \alpha) + \sum_{s'} Q(s, K)(s')V^i_C(s', K'(s, K)),
\]

and thus individual consumption (2) implies that:

\[
V^i_C(s, K, \alpha) = (\alpha_i)^{1/\sigma} V_C(s, K) + V_P(s, K)[\gamma_i - (\alpha_i)^{1/\sigma} \gamma],
\]

\[
= (\alpha_i)^{1/\sigma} [V_F(s, K) + V_w(s, K)I] + V_P(s, K)[\gamma_i - (\alpha_i)^{1/\sigma} \gamma].
\]

Also notice that \( W_i(s, K, \alpha) = V^i_C(s, K, \alpha) - V_w(s, K) \) and consequently it follows that:

\[
W_i(s, K, \alpha) = (\alpha_i)^{1/\sigma} V_F(s, K) + V_P(s, K)[\gamma_i - (\alpha_i)^{1/\sigma} \gamma] + V_w(s, K)[\gamma_i - (\alpha_i)^{1/\sigma} \gamma].
\]

Given some initial state \((s_0, K_0)\) and some initial distribution \(\theta^0\), we are ready to compute \(\alpha(s_0, K_0, \theta^0)\). Note that by definition

\[
W_i(s_0, K_0, \alpha(s_0, K_0, \theta^0)) = \theta^0_i [p(s_0, K_0) + d(s_0, K_0)],
\]

for all \(i\). Therefore, it follows that:

\[
\theta^0_i [p(s_0, K_0) + d(s_0, K_0)] = \theta^0_i V_F(s_0, K_0),
\]

\[
= (\alpha_i(s_0, K_0, \theta^0))^ {1/\sigma} V_F(s_0, K_0)
\]

\[
+ V_P(s_0, K_0)[\gamma_i - (\alpha_i(s_0, K_0, \theta^0))^ {1/\sigma} \gamma]
\]

\[
+ V_w(s_0, K_0)[\gamma_i - (\alpha_i(s_0, K_0, \theta^0))^ {1/\sigma} - 1],
\]

and consequently,

\[
(\alpha_i(s_0, K_0, \theta^0))^{1/\sigma} = \frac{\theta^0_i V_F(s_0, K_0) + V_w(s_0, K_0) - V_P(s_0, K_0)\gamma_i}{V_F(s_0, K_0) + V_w(s_0, K_0)I - V_P(s_0, K_0)\gamma_i}, \tag{21}
\]
for $i = 1, \ldots, (I - 1)$. Put $\alpha_I(s_0, K_0, \theta^0) = \left(1 - \sum_{j=1}^{I-1} (\alpha_j(s_0, K_0, \theta^0))^{1/\sigma}\right)^\sigma$. The assumption that the initial distribution of shares are "large enough" means that

$$\theta_i^0 V_F(s_0, K_0) + V_w(s_0, K_0) - V_P(s_0, K_0) \gamma_i > 0,$$

for all $i$. Finally, it can be checked as in Espino and Hintermaier [2005] that the candidate allocation corresponding to (21) and price system proposed constitute a RCE. \qed

**Proof of Proposition 4.** To see this, suppose that $F(K, L) = F(L)$ for all $K$ and $\delta = 0$. Thus, $y(s) = sF(I) + K_0$ represents the fruit delivered by this tree at state $s$ and $w(s) = sF'(I)$ is the wage per unit of labor. Since the Second Welfare Theorem holds, any Pareto optimal allocation (parametrized by $\alpha$), $c_i(\alpha) \in \mathbb{R}^S$, can be decentralized as a RCE with transfers. Equation (12) reduces to:

$$\phi_i(s, \alpha) = c_i(s, \alpha) - w(s) + \sum_{s'} Q(s, K)(s')\phi_i(s', \alpha).$$

for $s = 1, \ldots, S$. As pointed out by JKS [2003], this is a $S$-dimensional system with a unique solution continuous in $\alpha$. Continuity with respect to $\alpha$ implies that there exists $\alpha^0$ such that $\phi_i(s_0, \alpha^0) = [p(s_0) + d(s_0)] \theta_i^0$ for all $i$. Consequently, the resulting equilibrium portfolio in this economy is given by:

$$a'_i(s, \Phi)(s') = \phi_i(s', \alpha^0) - \theta_i^0 \left[p(s') + d(s')\right],$$

where $\Phi_i = \phi_i(s, \alpha^0)$ for each $i$. \qed

**Proof of Proposition 5.** The unique competitive equilibrium corresponding to $\alpha^0 = \alpha^0(s_0, K_0, \theta^0_i)$ defined by (21) will have equilibrium portfolios determined by:

$$A_i(s, K, \alpha^0) = \left[ (\alpha^0_i)^{1/\sigma} - \theta_i^0 \right] V_F(s, K) + [\gamma_i - (\alpha^0_i)^{1/\sigma} \gamma] V_P(s, K)$$

$$+ [I - (\alpha^0_i)^{1/\sigma} - 1] V_w(s, K).$$

25
If we assume that $\gamma_i = V_w(s, K) = 0$ for all $i$ and for all $(s, K)$, then it follows from (21) that $(\alpha_i^0)^{1/\sigma} = \theta_i^0$. Therefore, (22) implies that $A_i(s, K, \alpha^0) = 0$ for all $(s, K)$ and for all $i$. ■

**Proof of Proposition 6.** Suppose that $\gamma_i > 0$ for some agent $i$ and $V_w(s, K) = 0$ for all $(s, K)$. It follows from (22) that $A_i(s, K, \alpha^0) = [((\alpha_i^0)^{1/\sigma} - \theta_i^0)V_F(s, K) + [\gamma_i - (\alpha_i^0)^{1/\sigma} \gamma]V_P(s, K)$. The fixed equilibrium portfolio property means that $A_i(s, K, \alpha^0) = A_i(s, \alpha^0)$ for all $i$ and for all $(s, K)$. In particular, $A_i(s_0, K, \alpha^0) = 0$ for all $K$. This implies that $\alpha^0$ is independent of $K$. However under these assumptions, if we fix $s_t = s_0$ for all $t$, our framework reduces to the economy studied by Chatterjee [1994]. There he shows that the distribution of wealth (and consequently the distribution of consumption) depends on the initial level of capital (Theorem 3). This implies that the corresponding vector of welfare weights are not independent of the initial level of capital. Technical details are left to the interested reader. ■

**Proof of Proposition 7.** Suppose that $\gamma_i = 0$ for all $i$ and $V_w(s, K) > 0$ for all $(s, K)$. It follows from (22) that $A_i(s, K, \alpha^0) = [((\alpha_i^0)^{1/\sigma} - \theta_i^0)V_F(s, K) + [I (\alpha_i^0)^{1/\sigma} - 1]V_w(s, K)$. Again, the fixed equilibrium portfolio property means that $A_i(s, K, \alpha^0) = A_i(s, \alpha^0)$ for all $i$ and for all $(s, K)$. This implies that $\alpha^0$ is independent of $K$ and:

$$
\left( (\alpha_i^0)^{1/\sigma} - \theta_i^0 \right) V_F(s, K) = (1 - \theta_i^0 I) V_w (s, K),
$$

for all $(s, K)$. But this implies that human wealth and non-human wealth are not linearly independent and then we get a contradiction. ■

**References**


