Nonparametric Estimation of First-Price Auctions with Risk-Averse Bidders

Federico Zincenko*

Department of Economics
University of Pittsburgh

Incomplete Preliminary Draft
December 6, 2013

Abstract

In the context of a first-price sealed-bid auction model with risk-averse bidders, this paper proposes a nonparametric estimator for the bidders’ utility function and the density of private values. I adopt a fully nonparametric approach by not placing any restrictions on the shape of the bidders’ utility function beyond standard regularity conditions. The proposed estimator for the utility function is uniformly consistent, while the estimator of the density of private values attains Guerre, Perrigne, and Vuong (2000)’s optimal rate.

JEL Classification: C14, D44.
Keywords: First-price auction, risk aversion, independent private values, nonparametric estimation, sieve spaces.

*E-mail address: zincenko@pitt.edu. This article is developed from the first chapter of my Ph.D. dissertation at UCLA. I am grateful to Rosa L. Matzkin, Jinyong Hahn, and Ichiro Obara for their guidance and encouragement. For useful comments and detailed discussions, I thank Arie Beresteau, Zhipeng Liao, Enrique Seira, and Artyom Shneyerov. I am also grateful to the seminar participants at UC Los Angeles, ITAM, University of Pittsburgh, New Economic School (Moscow), Royal Holloway (University of London), and 2012 California Econometrics Conference (UC Davis). Financial support from the Dissertation Year Fellowship (UCLA) and the Welton Graduate Prize is gratefully acknowledged. I thank the CCPR at UCLA for its hospitality during the completion of this article.
1 Introduction

Risk aversion is essential to understanding economic decisions under uncertainty. In first-price sealed-bid auctions, risk aversion plays a fundamental role in explaining bidders’ behavior. Although several families of utility functions have been employed to describe different attitudes toward risk, in practice, we do not know which one accurately explains bidders’ behavior.

I consider a first-price sealed-bid auction with risk-averse bidders within the independent private values paradigm. In this setting, each potential buyer has his own private value for the item being sold and makes a sealed bid. The buyer who makes the highest bid wins the item and pays the seller the amount of that bid. This model is completely characterized by two objects. The first is the bidders’ utility function, which describes bidders’ risk preferences. The second is the density of private values, which describes the distribution of valuations for the auctioned item.

This paper develops an estimator that imposes no parametric specification on the common utility function of the bidders. Only standard regularity conditions are assumed. These assumptions are satisfied by linear (risk-neutral), constant relative risk aversion (CRRA), and constant absolute risk aversion (CARA) utility functions, as well as, many others. In this sense, my paper generalizes the empirical analysis of first-price auctions by nesting many existing estimators within a fully nonparametric framework.

This paper has two objectives. The first is to nonparametrically estimate the bidders’ utility function. Despite its relevance, only a few papers have proposed an estimator for such a function. Campo, Guerre, Perrigne, and Vuong (2011), for instance, adopt a semi-parametric approach and propose an estimator for the bidders’ risk aversion parameter. Their approach requires that the researcher imposes a parametric specification—such as CRRA or CARA—on the bidders’ utility function before estimating the risk aversion parameter and the density of private values. In real-world applications, the choice of the parametric specification may be arbitrary and not always realistic. In addition, there is no general agreement on which specification is the right one; when the choice is incorrect, the resulting estimator is invalid.

The second objective of my paper is to estimate the latent density of private values following a fully nonparametric perspective. To that end, I propose an estimator for the density of private values that does not rely on any parametric specification of the bidders’ utility function. The main advantage of this approach is that the resulting
estimator is robust to misspecification of such utility function. A common practice when estimating first-price auctions is to first assume a specific family of risk aversion for the bidders’ utility, and then, estimate the density of private values. This procedure has been justified so far because of its low implementation costs and the possibility of attaining the optimal global rate of convergence (Guerre et al. (2000) and Campo et al. (2011)). However, it can be criticized because an incorrect choice of the family of risk aversion invalidates the estimator for the density of private values.

Several papers have developed nonparametric estimators for the density of private values under the assumption that bidders are risk-neutral. The pioneering work of Guerre et al. (2000) constructed the first estimator to attain the optimal global rate of convergence. Recently, Marmer and Shneyerov (2012) have proposed an alternative estimator that is asymptotically normal and also attains the optimal rate. Bierens and Song (2012) have used integrated simulated moments to propose an estimator and construct a test for the validity of the first-price auction model. Here, I build on previous work by allowing bidders to be risk-averse.

My estimator for the density of private values is asymptotically normal, uniformly consistent, and attains Guerre et al. (2000)’s optimal rate. I derive these asymptotic properties by extending the approach of Marmer and Shneyerov (2012) to accommodate risk aversion from a nonparametric perspective. This has two advantages over existing work. First, empirical and experimental evidence indicates that risk aversion is a fundamental component of bidders’ behavior (see Guerre, Perrigne, and Vuong (2009), Section 1, as well as the references cited therein); therefore, invoking risk neutrality is likely to generate erroneous conclusions. Second, there is no evidence telling us which concept of risk aversion is the most appropriate to describe bidders’ risk preferences; therefore, it is essential to adopt a nonparametric approach.\footnote{Regarding the experimental evidence, I highlight the work of Delgado (2008), whose “findings are not inconsistent with a role for risk aversion in the tendency to bid too high.”}

To my knowledge, only two papers have analyzed the identification of the bidders’ utility function from a nonparametric perspective. Lu and Perrigne (2008) identified and estimated such a function by exploiting two auction designs, namely, ascending and first-price sealed-bid auctions. Guerre et al. (2009) improved on Lu and Perrigne (2008) and identified the bidders’ utility function by using the latter design only. They showed that the bidders’ utility function is nonparametrically identified under some exclusion restrictions. Their primary exclusion restriction was exogenous bidders’
participation. This exclusion restriction means that the distribution of valuations, or equivalently, the density of private values, is independent of the number of bidders. Exploiting this restriction, Guerre et al. (2009) developed their constructive identification strategy. However, such a strategy is recursive and based on an infinite series of differences in quantiles, so it does not lead to a natural estimator for the bidders’ utility function. Here, my contribution is to develop a valid estimator.

I develop a convenient identification procedure that allows us to estimate the objects of interest: the bidders’ utility function and the density of private values. Specifically, I characterize these objects by an argument that minimizes a certain functional over a space of smooth functions; in other words, the bidders’ utility function and the density of private values are characterized by an argument that minimizes certain functional. Such an argument is a smooth real-valued function and becomes the (infinite-dimensional) parameter of interest. I nonparametrically estimate this infinite-dimensional parameter by the method of sieve extremum estimation. This method optimizes an empirical criterion function over a sequence of finite-dimensional approximation spaces (sieve spaces); see Chen (2007). The estimators for the bidders’ utility function and the density of private values are smooth nonlinear functionals of the sieve estimator for the parameter of interest. In particular, the estimator for the utility function is uniformly consistent and preserves the basic properties of the utility function (strict monotonicity, concavity, and differentiability).

This paper is related to a vast literature on empirical industrial organization. First, it relates to the literature on structural econometrics of auction data. This literature is extensive and has expanded at an extraordinary rate; for example, see the surveys of Hendricks and Paarsch (1995), Laffont (1997), Perrigne and Vuong (1999), Athey and Haile (2007), and Hendricks and Porter (2007), as well as the textbook of Paarsch, Hong, and Haley (2006). I remark that nonparametric approaches have become very popular as auction data has become more available. Second, this paper is also related to the literature on recovering risk preferences from observed behavior. Within this line of research, I highlight the works of Lu (2004) and Ackerberg, Hirano, and Shahriari (2011). The former proposes a semiparametric method to estimate the risk aversion parameter, as well as the risk premium, in the context of a first-price sealed-bid auction with stochastic private values. The latter considers a buy price auction framework and nonparametrically identifies both time and risk preferences of the bidders.

The results obtained in this paper are relevant for public policy recommendations.
First-price sealed-bid auctions are used in many socio-economic contexts such as timber sales, outer continental shelf wildcat auctions (Li, Perrigne, and Vuong (2003)), and competitive sales of municipal bonds (Tang (2011)). In order to establish adequate auction rules that maximize the auctioneer’s revenue, we need robust information about bidders’ risk preferences; for example, the optimal reserve price depends on both the risk preferences and the distribution of valuations.

2 Auction Model and Data Generating Process

2.1 First-Price Auction Model

A single indivisible object is sold through a first-price sealed-bid auction with non-binding reserve price. In other words, the object is sold to the highest bidder who pays his bid to the seller and each bidder does not know others’ bids when forming his bid. Within the independent private value (IPV) paradigm, each bidder knows his own private value $v$, but not other bidders’ private values. There are $I \geq 2$ bidders and private values are drawn independently from a common cumulative distribution function (c.d.f.) $F(\cdot | I)$. Such a distribution is twice continuously differentiable with density $f(\cdot | I)$ and has compact support $[\underline{v} (I), \bar{v} (I)] \subseteq \mathbb{R}_{\geq 0}$. Both $I$ and $F(\cdot | I)$ are common knowledge.

All bidders are identical ex ante and the game is symmetric. Each bidder has the same univariate utility function $U(\cdot)$ that is independent of $I$. If a bidder with value $v$ wins and pays $b \geq 0$, his utility is $U(v - b)$, and if he loses, his utility is $U(0)$. Since any bidder’s payment must be smaller or equal than his own valuation, the domain of $U(\cdot)$ is restricted to $\mathbb{R}_{\geq 0}$. Bidder $i$ with valuation $v_i$ maximizes his expected utility $U(v_i - b_i) \Pr(b_i \geq b_j, j \neq i)$ with respect to his bid $b_i$, where $b_j$ is the $j$th-player’s bid. It is also assumed that $U(\cdot)$ is twice continuously differentiable, $U(0) = 0$, $U'(\cdot) > 0$, and $U''(\cdot) \leq 0$.

Only symmetric Bayesian Nash equilibria are considered. As a consequence, there exists a unique symmetric equilibrium bidding function $s(\cdot; I)$; see e.g. Hu, Matthews, and Zou (2010), Section 2, and the references cited therein. Such a function is strictly increasing, continuous on $[\underline{v} (I), \bar{v} (I)]$, and continuously differentiable on $(\underline{v} (I), \bar{v} (I)]$. 


Moreover, it satisfies the differential equation

\[ s'(v; I) = (I - 1) \frac{f(v|I)}{F(v|I)} \lambda_0(v - b) \tag{1} \]

with boundary condition \( s[\bar{v}(I); I] = \bar{v}(I) \), where \( b = s(v; I) \) is the optimal bid, \( s'(v; I) \) denotes the first derivative of \( s(v; I) \) with respect to \( v \), and \( \lambda_0(\cdot) \equiv U(\cdot)/U'(\cdot) \).

From equation (1), the equilibrium bidding function can also be written as

\[ s(v; I) = v - \lambda_0^{-1} \left[ \frac{s'(v; I)F(v|I)}{(I - 1)f(v|I)} \right], \]

where \( \lambda_0^{-1}(\cdot) \) stands for the inverse of \( \lambda_0(\cdot) \). Observe that the difference between a valuation and its bid depends crucially on both the bidders’ risk preferences and the distribution of valuations. It is further assumed that \( U(\cdot) \) and \( F(\cdot|I) \) satisfy standard regularity conditions: for a nonnegative integer \( R \), \([U(\cdot), F(\cdot|I)] \in U_R \times F_R \) where the sets of functions \( U_R \) and \( F_R \) are defined in Guerre et al. (2009), Section 2.1.

### 2.2 Data Generating Process

In practice, the auctioned object can be heterogeneous, so here I introduce an additional random vector \( X \) to account for the heterogeneity in the auctioned object. The econometrician observes a random sample \( \{(B_{pl}, I_l, X_l) : p = 1, \ldots, I_l, l = 1, \ldots, L\} \) where \( B_{pl} \) is the bid placed by the \( p \)th individual in the \( l \)th auction, \( I_l \) is the number of bidders in the \( l \)th auction, and \( X_l \) is a \( d \)-dimensional vector of continuous auction-specific covariates. The private values of the bidders \( \{V_{pl} : p = 1, \ldots, I_l; l = 1, \ldots, L\} \) are unobservable. Hereafter, I suppose that the data generating process satisfies the following assumption.

**Assumption 1.** The random vectors \( \{(V_{pl}, I_l, X_l) : p = 1, \ldots, I_l, l = 1, \ldots, L\} \) satisfy the following conditions.

1. \( \{(V_{1l}, \ldots, V_{Il}, I_l, X_l) : l = 1, \ldots, L\} \) are independent.

2. \( \{(I_l, X_l) : l = 1, 2, \ldots, L\} \) are identically distributed with joint density \( f_{IX}(\cdot, \cdot) \).
   Its support is \( \mathcal{I} \times \mathcal{X} \subset \mathbb{N}_{\geq 2} \times \mathbb{R}^d \) where \( 2 \leq \#(\mathcal{I}) < \infty \) and \( \mathcal{X} \) is compact with nonempty interior.
3. \( f_{IX}(\cdot, \cdot) \geq c_f > 0 \) on \( \mathcal{I} \times \mathcal{X} \), and for each \( i \in \mathcal{I} \), \( f_{IX}(i, \cdot) \) admits up to \( R + 1 \) continuously bounded partial derivatives on \( \mathcal{X} \).

4. For each \( l \), \( \{V_{pl}: p = 1, \ldots, I_l\} \) are independent and identically distributed conditionally on \((I_l, X_l)\) with density \( f_0(\cdot | X_l) = f_0(\cdot | I_l, X_l) \).

The fourth item imposes an exclusion restriction on the bidders’ participation, more precisely, the conditional density of private values must be conditionally independent of the number of bidders. As shown by Guerre et al. (2009), an exclusion restriction is necessary to identify an auction model with risk-averse bidders.

In addition to Assumption 1, I impose standard regularity conditions on the latent density \( f_0(\cdot | \cdot) \). It is assumed that \( f_0(\cdot | \cdot) \in \mathcal{F}_R^* \), where \( \mathcal{F}_R^* \) is defined as follows.

**Definition 1.** Let \( \mathcal{F}_R^* \) be the set of conditional densities \( f(\cdot | \cdot) \) satisfying the next requirements: \( f(\cdot | \cdot) \) has support \( S(F) \equiv \{ (v, x) : v \in [\underline{\psi}(x), \bar{\psi}(x)], x \in \mathcal{X} \} \) with \( 0 \leq \underline{\psi}(x) < \bar{\psi}(x) < M < \infty \); \( f(\cdot | \cdot) \geq c_f > 0 \) on \( S(F) \); and \( f(\cdot | \cdot) \) admits up to \( R \) continuous bounded derivatives on \( S(F) \).

The next assumption formalizes the idea that the bids are generated by a first-price auction with independent private values. It also establishes the smoothness of the bidders’ utility function and the conditional density of private values.

**Assumption 2.** The bids \( \{B_{pl}: p = 1, \ldots, I_l, l = 1, \ldots, L\} \) are generated by the auction model of Subsection 2.1 with structure \([U_0(\cdot), f_0(\cdot | \cdot)] \in \mathcal{U}_R \times \mathcal{F}_R^* \). To be specific, \( B_{pl} = s(V_{pl}; I_l, X_l) \) where, for each \((i, x) \in \mathcal{I} \times \mathcal{X} \), \( s(\cdot; i, x): [\underline{\psi}(x), \bar{\psi}(x)] \to \mathbb{R}_{\geq 0} \) satisfies the differential equation

\[
s(v; i, x) = v - \lambda_0^{-1} \left[ \frac{s'(v; i, x) F_0(v | x)}{(i - 1) f_0(v | x)} \right]
\]

with boundary condition \( s(\underline{\psi}(x); i, x) = \underline{\psi}(x) \), and \( \lambda_0^{-1}(\cdot) \) denotes the inverse of \( \lambda_0(\cdot) \equiv U_0(\cdot) / U_0'(\cdot) \).

The parameter of interest is the function \( \lambda_0^{-1}(\cdot) \), which does not depend on \( x \). Observe that both \( U_0(\cdot) \) and \( f_0(\cdot | \cdot) \) can be recovered from \( \lambda_0^{-1}(\cdot) \). The former is the (closed-form) solution of the differential equation \( \lambda_0(\cdot) U_0'(\cdot) - U_0(\cdot) = 0 \) with a boundary condition and a normalizing restriction, such as \( U(0) = 0 \) and \( U(1) = 1 \).
The conditional density of private values can be obtained from eq. (4) on Guerre et al. (2009). Specifically, we have that

\[ V_{pl} = B_{pl} + \lambda_0^{-1} \left[ \frac{1}{(i_l - 1)} \frac{G(B_{pl}|I_l, X_l)}{g(B_{pl}|I_l, X_l)} \right], \]  

(2)

where \( G(|I_l, X_l) \) and \( g(|I_l, X_l) \) denote the conditional c.d.f. and density of \( B_{pl} \) given \( (I_l, X_l) \), respectively.

The next proposition provides useful results for the rest of paper. Let define \( \mathcal{I}^* = \{(i, j) \in \mathcal{I}^2 : i < j\} \) and denote \( b(|i, x) : [0, 1] \rightarrow \mathbb{R}_{\geq 0} \) the conditional quantile function associated with \( G(|i, x) \).

**Proposition 1.** *Under Assumptions 1-2, the following statements hold.*

1. For each \( i \in \mathcal{I} \), the next properties are satisfied:
   
   (a) \( \inf \{ b(|i, x) - b(0|i, x) : x \in \mathcal{X} \} > 0 \) and \( b(\bar{0}|i, x) = \nu(x) \) for all \( x \in \mathcal{X} \).
   
   (b) \( G(|i, \cdot) \) admits \( R + 1 \) continuous partial derivatives on its support \( S_i(G) \equiv \{(b, x) : b \in [b(0|i, x), b(1|i, x)], x \in \mathcal{X} \} \).
   
   (c) \( g(|i, \cdot) \geq c_g > 0 \) on \( S_i(G) \) and admits \( R + 1 \) continuous partial derivatives on \( \tilde{S}_i(G) \equiv \{(b, x) : b \in (b(0|i, x), b(1|i, x)], x \in \mathcal{X} \} \).

2. For all \( (\alpha, x) \in (0, 1] \times \mathcal{X} \) and \( (i, j) \in \mathcal{I}^* \), \( b(\alpha|i, x) < b(\alpha|j, x) \).

3. (a) For all \( (\alpha, x) \in [0, 1] \times \mathcal{X} \) and \( (i, j) \in \mathcal{I}^* \), the compatibility condition

\[ b(\alpha|j, x) - b(\alpha|i, x) = \lambda_0^{-1}[R(\alpha|i, x)] - \lambda_0^{-1}[R(\alpha|j, x)] \]

(3)

is satisfied, where \( R(\alpha|i, x) \equiv \alpha b'(\alpha|i, x)/(i - 1) \) and \( b'(\alpha|i, x) \) is the derivative of \( b(\alpha|i, x) \) with respect to \( \alpha \), i.e., the conditional quantile density.

(b) For each \( i \in \mathcal{I} \), the function \( \xi_i : S_i(G) \rightarrow \mathbb{R}_{\geq 0} \) defined by

\[ \xi_i(b, x) = b + \lambda_0^{-1} \left[ \frac{1}{(i - 1)} \frac{G(b|i, x)}{g(b|i, x)} \right] \]

satisfies \( \partial \xi_i(b, x)/\partial b > 0 \) for every \( (b, x) \in S_i(G) \).
The first item follows immediately from Lemma 1 in Campo et al. (2011). The second and third items are extensions of Lemma 1 in Guerre et al. (2009) to the case of exogenous variables. To conclude this section, the next lemma establishes a crucial property of the function $R(|\cdot|, \cdot)$. Its proof is detailed in the Appendix A.

**Lemma 1.** Under Assumptions 1-2, there exist $c_R > 0$ and $\tilde{\alpha} \in (0, 1]$ such that

$$R'(\alpha |j|, x) \geq c_R \quad \text{and} \quad R'(\alpha |i|, x) - R'(\alpha |j|, x) \geq c_R$$

for all $(\alpha, x) \in [0, \tilde{\alpha}] \times \mathcal{X}$ and $(i, j) \in \mathcal{I}^*$, where $R'(\alpha |i|, x) = \partial R(\alpha |i|, x)/\partial \alpha$.

### 3 Nonparametric Estimation

The purpose of this section is to build a nonparametric estimator for $\lambda_{\theta}^{-1}(\cdot)$, the parameter of interest. As a measure of distance, we consider the sup-norm over the interval $[0, \bar{u}]$ where $\bar{u} = \max\{R(\alpha |i_1|, \bar{x}) : \alpha \in [0, \tilde{\alpha}]\}, \ i_1 = \min\{\mathcal{I}\}$, and $(\tilde{\alpha}, \bar{x}) \in (0, 1) \times \text{int} (\mathcal{X})$ is arbitrarily chosen by the researcher. The parameter space, which contains $\lambda_{\theta}^{-1} : [0, \bar{u}] \to \mathbb{R}_{\geq 0}$, is defined as follows.

**Definition 2.** Let $\mathcal{H}_R$ be the space of functions $\phi : [0, \bar{u}] \to \mathbb{R}_{\geq 0}$ that satisfy the next conditions: $\phi(0) = 0$, $\phi(\cdot)$ admits $R + 1$ continuous derivatives on $[0, \bar{u}]$, and $0 \leq \phi'(\cdot) \leq 1$.

For the remaining discussion, I will use the following notation. The sup-norm of function $f(\cdot)$ over a set $Z$ is denoted by $\|f\|_{\infty, Z} = \sup_{z \in Z} |f(z)|$. The indicator function is denoted by $1\{\cdot\}$, $f^{(r)}$ stands for the $r$-th derivative of $f$, and $f^{(0)} = f$.

### 3.1 Identification Strategy

In this subsection, I propose a population criterion function that allows us to identify $\lambda_{\theta}^{-1}(\cdot)$ within the parameter space, $\mathcal{H}_R$. Such a criterion function $Q_\varepsilon : \mathcal{H}_R \to \mathbb{R}_{\geq 0}$ is defined by

$$Q_\varepsilon(\phi) = \max_{a \in [\varepsilon, \tilde{\alpha}]} |b(a|i_2, \bar{x}) - b(a|i_1, \bar{x}) + \phi[R(a|i_2, \bar{x})] - \phi[R(a|i_1, \bar{x})]|,$$

where $\varepsilon \in [0, \tilde{\alpha})$, $i_2 \in \mathcal{I}$, and $i_2 > i_1$. The functional form of $Q_\varepsilon(\cdot)$ is based on the compatibility condition discussed in Proposition 1.3(a).
It is clear that $\lambda_0^{-1}(\cdot)$ is an argument that minimizes $Q_\varepsilon(\cdot)$ for any $\varepsilon \in [0, 1)$; note that $Q_\varepsilon(\lambda_0^{-1}) = 0$ and $Q_\varepsilon(\cdot) \geq 0$. The next proposition establishes the uniqueness of $\lambda_0^{-1}(\cdot)$ provided that $\varepsilon > 0$ is sufficiently small. In other words, $\lambda_0^{-1}(\cdot)$ is characterized as the unique argument that minimizes $Q_\varepsilon(\cdot)$ over $\mathcal{H}_R$.

**Proposition 2.** Under Assumptions 1-2, there are constants $\varepsilon \in (0, 1]$ and $K > 0$ such that the following implication holds. For any $\phi \in \mathcal{H}_R$ and $\varepsilon \in [0, \varepsilon)$,

$$\|\phi - \lambda_0^{-1}\|_{[0,u],\infty} \geq \varepsilon \implies Q_\varepsilon(\phi) \geq K\varepsilon/\log(\varepsilon^{-1}).$$

Furthermore, this implication cannot be improved. To be precise, for any $\varepsilon' \in (0, 1]$ and $K' > 0$, there are $\varepsilon \in (0, \varepsilon')$ and $\phi \in \mathcal{H}_R(M)$ that satisfy

$$\|\phi - \lambda_0^{-1}\|_{[0,u],\infty} \geq \varepsilon \& Q_\varepsilon(\phi) \leq K'\varepsilon/\log(\varepsilon^{-1}).$$

### 3.2 Preliminary Kernel Estimators

This subsection constructs nonparametric estimators for the functions $b(\cdot;\cdot)$ and $R(\cdot;\cdot)$. Following recent literature, I employ a kernel approach. The resulting estimators will be used later to compute the empirical counterpart of $Q_\varepsilon(\cdot)$.

Let $k(\cdot)$ be a univariate kernel, and also, let $h_g$ and $h_X$ be bandwidths. I make the following assumptions about these objects.

**Assumption 3.** The kernel $k(\cdot)$ is symmetric with $R + 1$ continuous derivatives on $\mathbb{R}$, has support $[-1, 1]$, satisfies $\int k(v)dv = 1$, and its order is $R + 1$.

**Assumption 4.** Let $\gamma_g$ and $\gamma_Q$ be positive constants. The bandwidths are of the form: $h_g = \gamma_g[\log(L)/L]^{1/(2R+d+3)}$ and $h_X = \gamma_X[\log(L)/L]^{1/(2R+d+2)}$. 

10
The estimators of \( f_X(\cdot), f_{IX}(\cdot, \cdot), g(\cdot, \cdot, \cdot), \) and \( G(\cdot, \cdot) \) are defined as follows:

\[
\hat{f}_X(x) = \frac{1}{Lh_X} \sum_{i=1}^{L} K \left( \frac{x-X_i}{h_X} \right),
\]

\[
\hat{f}_{IX}(i, x) = \frac{1}{Lh_X} \sum_{i=1}^{L} \mathbb{1}\{I_i = i\} K \left( \frac{x-X_i}{h_X} \right),
\]

\[
\hat{g}(b, i, x) = \frac{1}{Lh_g} \sum_{i=1}^{L} \sum_{j=1}^{I_i} \mathbb{1}\{I_i = i\} k \left( \frac{b-B_{pl}}{h_g} \right) K \left( \frac{x-X_i}{h_g} \right),
\]

\[
\hat{G}(b|i, x) = \frac{1}{\hat{f}_{IX}(i, x) Lh_X} \sum_{i=1}^{L} \sum_{j=1}^{I_i} \mathbb{1}\{B_{pl} \leq b, I_i = i\} K \left( \frac{x-X_i}{h_X} \right),
\]

where \((b, i, x) \in \mathbb{R}_{\geq 0} \times \mathcal{I} \times \mathcal{X}\) and \(K(\cdot)\) is the product kernel, i.e., \(K(x) = \prod_{j=1}^{d} k(x_j)\). The shape of these estimators will allow us to build an asymptotically normal estimator for the density of private values. Moreover, the order of \(k(\cdot)\) has been chosen according to the smoothness of \(G(\cdot, i, x)\).

The conditional quantile function \(b(\alpha|i, x)\) and its derivative with respect to \(\alpha\), \(b'(\alpha|i, x)\), is estimated as in Marmer and Shneyerov (2012). To be specific,

\[
\hat{b}(\alpha|i, x) = \inf \{b \in \mathbb{R}_{\geq 0} : \hat{G}(b|i, x) \geq \alpha\}
\]

and \(\hat{b}'(\alpha|i, x) = 1/\hat{g}(\hat{b}(\alpha|i, x)|i, x]\). Naturally, \(\hat{R}(\alpha|i, x) = \alpha \hat{b}'(\alpha|i, x)/(i-1)\) becomes the estimator of \(R(\alpha|i, x)\).

### 3.3 The Estimator: Definition and Uniform Consistency

From the discussion of Subsection 3.1, the parameter of interest can be characterized as the unique argument that minimizes \(Q_\varepsilon(\cdot)\) over \(\mathcal{H}_R\). Formally,

\[
\lambda^{-1}_0(\cdot) = \arg \min_{\phi(\cdot) \in \mathcal{H}_R} Q_\varepsilon(\phi)
\]

provided that \(\varepsilon > 0\) is sufficiently small. Given this characterization, I construct the estimator of \(\lambda^{-1}_0(\cdot)\) as the argument that minimizes the empirical counterpart of \(Q_\varepsilon(\cdot)\) over a sieve space, i.e., a finite-dimensional approximation space.

As a starting point, I construct the empirical criterion function \(\hat{Q}(\cdot)\), which is the
empirical counterpart of $Q_{h_Q}(\cdot)$. Specifically,

$$
\hat{Q}(\phi) = \max_{a \in [h_Q, \bar{\alpha}]} \left[ \hat{b}(a|\bar{x}_2) - \hat{b}(a|\bar{x}_1) + \phi[\hat{R}(a|\bar{x}_2)] - \phi[\hat{R}(a|\bar{x}_1)] \right],
$$

where $h_Q$ is of exact order $1/\psi^{-1}\left[\frac{L}{\log(L)}\right]^{(R+1)/(2R+d+3)}$ and $\psi(x) \equiv x \log(x)$. The next lemma states that $\hat{Q}(\cdot)$ converges uniformly in probability to $Q_{r_L}(\cdot)$, i.e., its population counterpart.

**Lemma 2.** Under Assumptions 1-4, $\sup_{\phi \in \mathcal{H}_R} |\hat{Q}(\phi) - Q_{r_L}(\phi)| = O_P(h_{gR+1}).$

A natural way to define the estimator of $\lambda_0^{-1}(\cdot)$ is

$$
\hat{\lambda}^{-1}(\cdot) = \arg \min_{\phi(\cdot) \in \mathcal{H}(L)} \hat{Q}(\phi),
$$

where $\mathcal{H}(L) \subset \mathcal{H}_R$ is a sieve space whose dimension increases with $L$ at an appropriate rate. The next theorem establishes the uniform consistency of $\hat{\lambda}^{-1}(\cdot)$ with its rate of convergence.

**Theorem 1** (Uniform Consistency). Let $\{\mathcal{H}(L) \subset \mathcal{H}_R : L \in \mathbb{N}\}$ be a sequence of sieve spaces that satisfies

$$
\left[ \frac{L}{\log(L)} \right]^{\frac{R+1}{2R+d+3}} \inf_{\phi(\cdot) \in \mathcal{H}(L)} \|\phi(\cdot) - \lambda_0^{-1}(\cdot)\|_{[0,\bar{u}],\infty} = O(1),
$$

as $L \to \infty$. Under Assumptions 1-4,

$$
\psi^{-1}\left[\frac{L}{\log(L)}\right]^{\frac{R+1}{2R+d+3}} \|\hat{\lambda}^{-1}(\cdot) - \lambda_0^{-1}(\cdot)\|_{[0,\bar{u}],\infty} = O_P(1)
$$

with $\psi(x) \equiv x \log(x)$.

Note that $\psi^{-1}(x) < x$ for all $x > \exp(1)$, therefore, the rate of convergence of $\hat{\lambda}^{-1}(\cdot)$ is slower than Stone (1982)’s uniform rate, which is $\left[ \frac{L}{\log(L)} \right]^{(R+1)/(2R+d+3)}$. However, for any fixed $c \in (0, 1)$, we have that $\psi^{-1}(x) > x^c$ when $x > 0$ is sufficiently large. Hence, Theorem 1’s rate can be arbitrarily close to Stone (1982)’s rate up to a exponent.
A Appendix: Proofs

This appendix provides the proofs of the lemmas, propositions, and theorems stated in the body of the text. Proofs of the auxiliary lemmas are given in Appendix B.

A.1 Proof of Lemma 1

Pick any \((i, j) \in \mathcal{I}^*\). For the first statement, which is \(R'(\alpha|j, x) \geq c_R\), note that

\[
R'(\alpha|j, x) = \frac{1}{j - 1}[b'(\alpha|j, x) - \alpha b''(\alpha|j, x)].
\]

Since \(b'(\cdot, \cdot)\) is bounded away from zero on \([0, 1] \times \mathcal{X}\) and \(|b''(\cdot|j, \cdot)|\) is bounded above on \([0, 1] \times \mathcal{X}\), we can pick

\[
\tilde{\alpha} = \min_{\alpha, x} \{b'(\alpha|j, x) : (\alpha, x) \in [0, \tilde{\alpha}] \times \mathcal{X}\}
\]

Then, we can write

\[
R'(\alpha|i, x) - R'(\alpha|j, x) = \tilde{\delta}'(\alpha, x) + \tilde{\delta}(\alpha, x),
\]

where \(\tilde{\delta}'(\cdot, \cdot)\) stands for the partial derivative of \(\tilde{\delta}(\cdot, \cdot)\) with respect to its first argument. Since \(|\tilde{\delta}'(\cdot, \cdot)|\) is bounded above on \([0, 1] \times \mathcal{X}\), it suffices to show that

\[
\min_{\alpha, x} \{\tilde{\delta}(\alpha, x) : (\alpha, x) \in [0, \tilde{\alpha}] \times \mathcal{X}\} \geq c_R
\]

for some fixed \(\tilde{\alpha} \in (0, 1)\) and \(c_R > 0\). To do so, observe that

\[
\tilde{\delta}(0, x) = \left[\frac{\lambda_0(0)}{(i - 1)\lambda_0'(0) + 1} - \frac{\lambda_0'(0)}{(j - 1)\lambda_0''(0) + 1}\right] \frac{1}{f_0[\hat{\nu}(x)|x]} > 0
\]

for all \(x \in \mathcal{X}\). So define the function \(\Delta(\alpha) = \min_{x \in \mathcal{X}} \tilde{\delta}(\alpha, x)\), which is continuous (see e.g., Theorem 3.6 in Stokey, Lucas, and Prescott (1989)) and satisfies \(\Delta(0) > 0\)
because $\mathcal{X}$ is compact. By continuity, there is $\tilde{\alpha} > 0$ such that $\Delta(\alpha) \geq \Delta(0)/2 > 0$ for all $\alpha \in [0, \tilde{\alpha}]$, and therefore, $\tilde{\delta}(\alpha, x) > 0$ for all $(\alpha, x) \in [0, \tilde{\alpha}] \times \mathcal{X}$. The inequality (A.1) emerges from the compactness of $[0, \tilde{\alpha}] \times \mathcal{X}$.

**A.2 Proof of Proposition 2**

Before staring with the proof, I state the next auxiliary lemma.

**Lemma A.1.** For each $(u, x) \in [0, \tilde{u}] \times \mathcal{X}$, define recursively the following sequence:

$$
\alpha_0(u, x) = \min\{\alpha \in [0, \tilde{\alpha}] : R(\alpha|i, x) = u\}
$$

and

$$
\alpha_t(u, x) = \min\{\alpha \in [0, 1] : R(\alpha|i, x) = R[\alpha_{t-1}(x)]|j, x]\}
$$

for $t \in \mathbb{N}$. Under Assumptions 1-2, the following statements hold.

1. There exists a finite $\tilde{T} \in \mathbb{N}$, independent of $u$ and $x$, such that $\alpha_t(u, x) \leq \tilde{\alpha}$ for all $t \geq \tilde{T}$ and $(u, x) \in [0, \tilde{u}] \times \mathcal{X}$.

2. Define $T_\varepsilon(u, x) = \min\{t \in \mathbb{N} : \alpha_t(u, x) \leq \varepsilon\}$ where $\varepsilon > 0$ is sufficiently small. Then, $T_\varepsilon(u, x) \leq 2\log(\varepsilon^{-1})$ for all $(u, x) \in [0, \tilde{u}] \times \mathcal{X}$.

Without loss of generality, assume that $\mathcal{I} = \{i, j\}$ with $i < j$. Choose any $\varepsilon > 0$ small enough, and then, pick any $\phi \in \mathcal{H}_R$ such that $\|\phi - \lambda_0^{-1}\|_{[0, \tilde{u}], \infty} \geq \varepsilon$. By Proposition 1, we have that

$$b(\alpha|j, x) - b(\alpha|i, x) = \lambda_0^{-1}[R(\alpha|i, x)] - \lambda_0^{-1}[R(\alpha|j, x)].$$

for all $x \in \mathcal{X}$, so the criterion function becomes

$$Q_\varepsilon(\phi) = \max_{\alpha \in [\varepsilon, \tilde{\alpha}]} |\tilde{\phi}[R(\alpha|i, x)] - \tilde{\phi}[R(\alpha|j, x)]|$$

with $\tilde{\phi}(u) \equiv \phi(u) - \lambda_0^{-1}(u)$.

On the one hand, suppose that $|\tilde{\phi}'(0)| \geq \varepsilon^{1/3}$. For all $x \in \mathcal{X}$, then we have that

$$|\tilde{\phi}[R(\varepsilon^{1/2}|i, x)] - \tilde{\phi}[R(\varepsilon^{1/2}|j, x)]| = |\tilde{\phi}'(u^*_\varepsilon)|R(\varepsilon^{1/2}|i, x) - R(\varepsilon^{1/2}|j, x)|$$

$$= |\tilde{\phi}'(u^*_\varepsilon)|\tilde{\delta}(\varepsilon^{1/2}, x)|\varepsilon^{1/2}$$

$$\geq c_\delta|\tilde{\phi}'(u^*_\varepsilon)|\varepsilon^{1/2}$$

14
where \( u^*_x \in [R(\varepsilon^{1/2}|j, x), R(\varepsilon^{1/2}|i, x)] \) and \( \tilde{\phi}(\cdot, \cdot) \) has been defined in Subsection A.1. By continuity of \( \tilde{\phi}'(\cdot) \) and since \( \varepsilon > 0 \) was chosen small enough, it follows that \( |\tilde{\phi}'(u^*_x)| \geq \varepsilon^{1/2} \), which implies \( Q(\phi) \geq c_0 \delta \varepsilon \).

On the other hand, suppose that \( |\tilde{\phi}'(0)| < \varepsilon^{1/3} \). By continuity and since \( \tilde{\phi}(0) = 0 \), there is \( u^* \in (0, \bar{u}) \) such that \( |\tilde{\phi}(u^*)| = \varepsilon \). Then, for each \( x \in \mathcal{X} \), consider the sequence \([\alpha_t(u^*, x)]_{t \in \mathbb{N}}\) defined by eq. (A.2). By standard triangular inequalities, we have that

\[
|\tilde{\phi}(u^*)| \leq \sum_{t=0}^{T_\varepsilon(u, x) - 1} |\tilde{\phi}(R[\alpha_t(u^*, x)|i, x]) - \tilde{\phi}(R[\alpha_t(u^*, x)|j, x])| + |\tilde{\phi}(R[\alpha_{T_\varepsilon(u, x)}(u^*, x)|i, x])|
\]

where \( T_\varepsilon(u, x) \) has been defined in Lemma A.1.2. As \( |\tilde{\phi}'(0)| < \varepsilon^{1/3} \), it follows that \( |\tilde{\phi}(R[\alpha_{T_\varepsilon(u, x)}(u^*, x)|i, x])| < \varepsilon/2 \), and as a result,

\[
\frac{\varepsilon}{2} \leq 2[\log(\varepsilon^{-1})] \| \tilde{\phi}(R[\cdot|i, x]) - \tilde{\phi}(R[\cdot|j, x]) \|_{[\varepsilon, \bar{\alpha}], \infty}.
\]

**B Appendix: Proofs of Auxiliary Lemmas**

**References**


